International Journal of Mechanics and Thermodynamics.
ISSN 2278-361X Volume 13, Number 1 (2022), pp. 1-13

http://www.irphouse.com

Application of Chebyshev Wavelets to Ordinary Differential Equations

Lakshmi B. N.[1], Sumana K. P.[2], Asha C. S.[3], Achala L. N.[4]

[1,2,3,4] *P. G. Department of Mathematics and Research Centre in Applied Mathematics, M E S College of Arts, Commerce and Science, Malleswaram, Bengaluru, Karnataka -560003, India.*

Abstract

In this paper, we have solved linear and nonlinear, initial and boundary value problems using Chebyshev wavelet collocation method. In this method, the differential equations are converted to system of linear algebraic equations which can be easily evaluated. Quasilinearization technique is used to handle the nonlinear terms arising in the differential equations. The results obtained are compared with the exact solution. We observe that the Chebyshev wavelet solutions agree with the exact solutions and the same has been depicted through tables and graphs.

Keywords: Wavelets, Chebyshev wavelets, Differential equations, Quasilinearization, Collocation points.

1 INTRODUCTION

Wavelet is a trending new method to solve the difficult problems emerging in the field of mathematics, physics and engineering. The concept of 'ondelettes' or 'wavelets' originated from the study of time-frequency signal analysis, wave propagation, and sampling theory [1, 2]. In 1909, the 'wavelets' were first studied in a thesis by Alfred Haar and then the theoretical form was first proposed by Jean Morlet et al. [3] in the Marseille Theoretical Physics Center. Later, in 1988, wavelet analysis has been developed mainly by Y. Meyer and S. Mallat [4].

Wavelets are powerful mathematical tools which are considered abundantly to obtain accurate solutions of integral and differential equations [6]. Wavelets have gained popularity because of their ability to study the functions at different scale features. We come across different wavelet families applied to different studies like for example

Haar, Daubechies, Chebyshev, Legendre and B-Spline wavelets. Chebyshev wavelet is constructed from Chebyshev polynomial as their basis functions. They have excellent interpolating property and gives better accuracy for numerical approximations [5].

Wavelets have many application in signal and image processing like compression, denoising, discontinuity detection, audio enhancement and effects, edge detection, image fusion, image enhancement and many other applications. Recently, the methods based on the orthogonal functions and polynomials series, including wavelets are being used to approximate the solution of various problems. The main advantage of using orthogonal basis is that, it reduces these problems into systems of algebraic equations [5].

Awashie et al. [5] applied the Chebyshev wavelets for simulating a two-phase flow of immiscible fluids in a reservoir with different capillary effect. They observed that the discontinuities exist in the two-phase flow which are caused by the capillary pressure as expected physically. Adibi et al. [6] obtained the numerical solution of Fredholm integral equations of the first kind using Chebyshev wavelets. Hosseini et al. [7] used Chebyshev wavelet collocation method to solve ordinary differential equations. They tested spectral method for the same work which doesn't work well for ordinary differential equations. They also applied the Chebyshev wavelet Galerkin method for these kind of problems.

Celik [8] applied Chebyshev wavelet collocation method to determine the solution of linear ordinary differential equations of second order. They also obtained the approximate solution of Bessel differential equation of order zero and the Lane-Emden equation using the same method. Celik [2] also solved a class of linear and nonlinear nonlocal boundary value problems of second and fourth order using Chebyshev wavelet collocation method. Shiralashetti et al. [9] employed Chebyshev wavelet collocation method to obtain the solution of linear and nonlinear ordinary differential equations. Heydari et al. [10] obtained the solution of partial differential equations using Chebyshev wavelet collocation method.

Oruc et al. [11] considered the one dimensional time dependent coupled Burger's equation along with the suitable initial and boundary conditions. The numerical solutions of this equation is obtained by using Chebyshev wavelet collocation method.Usman et al. [12] considered MHD 3-D fluid flow in the presence of slip and thermal radiation effects and solved using Chebyshev wavelets. They observed that a suitable selection of stretching ratio parameter will help in hastening the heat transfer rate for a fixed value of velocity slip parameter and in reducing the viscous drag over the stretching sheet. Also efficiency of the method was shown by convergent analysis.

In order to solve single or systems of nonlinear ordinary and partial differential equations, Bellman and Kalaba [13] introduced the quasilinearization approach as a generalization of the Newton-Raphson method. This technique has quadratic rate of convergence [14, 15]. In this paper, we have considered the Chebyshev wavelet collocation method along with quasilinearization technique to obtain the numerical solution of linear and nonlinear, initial and boundary value problems.

2 CHEBYSHEV WAVELETS

Wavelets constitute a family of functions constructed from dilation and translation of a single functions called as the Mother wavelet. If this dilation parameter a and translation parameter b are allowed to vary continuously then, we get a family of continuous wavelets [11],

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\ \psi\left(\frac{x-b}{a}\right),$$

where $a, b \in \mathbb{R}, a \neq 0$.

The family of Chebyshev wavelets [2] are defined in the interval $[0, 1)$ as,

$$C_i(x) = C_{nm}(x) = \begin{Bmatrix} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0\,, & \text{otherwise} \end{Bmatrix} \ldots\ldots\ldots\ldots (1)$$

where

$$\alpha_m = \begin{Bmatrix} \sqrt{2}, & m = 0 \\ 2, & \text{otherwise} \end{Bmatrix}$$

where $i = n + 2^{k-1}m, k$ is any positive integer, $n = 1,2, \ldots, 2^{k-1}, m = 0,1,2, \ldots, M - 1, M$ is the maximum degree of Chebyshev wavelets of first kind and x is the normalized time. $T_m(x)$ are Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}\ on\ [-1, 1]$. The Chebyshev polynomials satisfy the following recurrence formula,

$$T_0(x) = 1, T_1(x) = x, T_{m+1} = 2xT_m(x) - T_{m-1}(x), \qquad \forall\ m = 1,2,3, \ldots.$$

The wavelet collocation points are defined as

$$x_j = \frac{j - 0.5}{N}, \qquad \forall\ j = 1,2, \ldots, N,$$

where $N = 2^{k-1}M$.

In order to solve the differential equations of second order, we require the following integrals.

$$P_i(x) = \int_0^x C_i(x)dx \ \ and \ \ Q_i(x) = \int_0^x P_i(x)dx\,.$$

2.1 FUNCTION APPROXIMATION

A function $f(x)$ which is square integrable on [0,1) can be expressed as infinite sum of Chebyshev wavelets as [5],

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}C_{nm}(x) \ldots\ldots\ldots\ldots (2)$$

where

$$a_{nm} = \int_0^1 f(x) C_{nm}(x) \omega_n(x) dx \ldots \ldots \ldots \ldots (3)$$

If the function $f(x)$ is approximated as piecewise constant in each subinterval, then equation (2) becomes

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm} C_{nm}(x) \ldots \ldots \ldots \ldots (4)$$

where a_{nm} are the Chebyshev wavelet coefficients to be determined.

3 METHOD OF SOLUTION

In this section, the Chebyshev wavelet collocation method (CWCM) is used to solve the linear and nonlinear, initial and boundary value problems.

Example 1:

Consider the linear ordinary differential equation of the form,

$$y'' - 3y' + 2y = 6e^{-x} \ldots\ldots\ldots\ldots (5)$$

with initial conditions

$$y(0) = 2 , y'(0) = 2 \ldots\ldots\ldots\ldots (6)$$

and the exact solution is $y(x) = 2e^{2x} - e^{x} - e^{-x}$.

The Chebyshev wavelet solution of the form,

$$y''(x) = \sum_{i=1}^{N} a_i \, C_i(x) \ldots \ldots \ldots \ldots (7)$$

where $a_i's,\ i = 1,2, \ldots, N$ are Chebyshev wavelet coefficients to be determined.

Integrating equation (7) twice with respect to x from 0 to x and using equation (6), we get

$$y'(x) = 2 + \sum_{i=1}^{N} a_i \, P_i(x) \ldots \ldots \ldots \ldots (8)$$

$$y(x) = 2 + 2x + \sum_{i=1}^{N} a_i \, Q_i(x) \ldots \ldots \ldots \ldots (9)$$

Substituting equations (7), (8) and (9) in (5), we obtain

$$\sum_{i=1}^{N} a_i\,(C_i(x) - 3P_i(x) + 2Q_i(x)) = 6e^{-x} - 4x + 2 \;\ldots\ldots\ldots\ldots\; (10)$$

Taking the collocation points $x = x_j$ in equations (10) and (9), we have

$$\sum_{i=1}^{N} a_i\,(C_i(x_j) - 3P_i(x_j) + 2Q_i(x_j)) = 6e^{-x_j} - 4x_j + 2 \ldots\ldots\ldots\ldots\; (11)$$

$$y(x_j) = 2 + 2x_j + \sum_{i=1}^{N} a_i\, Q_i(x_j) \ldots\ldots\ldots\ldots\; (12)$$

The wavelet coefficients $a_i,\ i = 1,2,\ldots,N$ are obtained by solving the N system of equations obtained in equation (11). These coefficients are then substituted in equation (12) to obtain the Chebyshev wavelet solutions at the collocation points $x_j, j = 1,2,\ldots,N$. Table 1 shows the comparison of CWCM and exact solution. Figure 1 depicts the typical behaviour of the solution for $k = 3, M = 10$.

EXAMPLE 2:

Consider the nonlinear ordinary differential equation with initial conditions

$$y'' + y' + y + y^2\, y' = -sin\, x - sin\, x \cos^2 x \ldots\ldots\ldots\ldots\; (13)$$

$$y(0) = 1\,, y'(0) = 0 \ldots\ldots\ldots\ldots\; (14)$$

The exact solution is $y(x) = cos\ x$.

As illustrated in example 1, we have

$$y''(x) = \sum_{i=1}^{N} a_i\, C_i(x) \;\ldots\ldots\ldots\ldots\; (15)$$

$$y' = \sum_{i=1}^{N} a_i\, P_i(x) \;\ldots\ldots\ldots\ldots\; (16)$$

$$y(x) = 1 + \sum_{i=1}^{N} a_i\, Q_i(x) \;\ldots\ldots\ldots\ldots\; (17)$$

where $a_i's,, i = 1,2,\ldots,N$ are Chebyshev wavelet coefficients to be determined. Using quasilinearization technique to handle the nonlinear terms in equation (13), we get

$$y''_{r+1} + (1 + y_r^2)y'_{r+1} + (1 + 2y_r y'_r)y_{r+1} = 2y_r^2\, y'_r - \sin x - \sin x\, cos^2 x$$

$$\ldots\ldots\ldots\ldots\; (18)$$

Substituting equations (15), (16) and (17) in equation (18) and taking the collocation points $x = x_j$, we obtain

$$\sum_{i=1}^{N} a_i \left(C_i(x_j) + (1 + y_r^2) P_i(x_j) + (1 + 2y_r y_r') Q_i(x_j) \right) =$$

$$2y_r^2\, y_r' - \sin x_j - \sin x_j\, cos^2 x_j - (1 + 2y_r y_r') \dots\dots\dots\dots (19)$$

$$y(x_j) = 1 + \sum_{i=1}^{N} a_i\, Q_i(x_j) \dots\dots\dots\dots (20)$$

Further, equation (19) is solved to get the wavelet coefficients $a_i, i = 1,2,\dots,N$ which are used to obtain the Chebyshev wavelet solutions at the collocation points $x_j, j = 1,2,\dots,N$ from equation (20). For $k = 3, M = 10$ Table 2 represents the comparison of CWCM and the exact solution. The approximate solution and the exact solution are plotted in

Figure **2**.

EXAMPLE 3:

Consider the linear ordinary differential equation with boundary conditions

$$y'' + y = 1 \dots\dots\dots\dots (21)$$

$$y(0) = 0\,, y(1) = 1 \dots\dots\dots\dots (22)$$

and the exact solution is given by $y(x) = -cos\ x + cot1\ sin\ x + 1$.

The Chebyshev wavelet solution is of the form,

$$y''(x) = \sum_{i=1}^{N} a_i\, C_i(x) \dots\dots\dots\dots (23)$$

where $a_i's,\ i = 1,2,\dots,N$ are Chebyshev wavelet coefficients to be determined.

Integrating equation (23) twice with respect to x from 0 to x and using equation (22), we get

$$y'(x) = y'(0) + \sum_{i=1}^{N} a_i\, P_i(x) \dots\dots\dots\dots (24)$$

$$y(x) = xy'(0) + \sum_{i=1}^{N} a_i\, Q_i(x) \dots\dots\dots\dots (25)$$

Putting $x = 1$ in equation (25) and using equation (22), we obtain

$$y'(0) = 1 - \sum_{i=1}^{N} a_i\, Q_i(1) \dots\dots\dots\dots (26)$$

Substituting equation (26) in equations (24) and (25), we get

$$y'(x) = 1 + \sum_{i=1}^{N} a_i\, (P_i(x) - Q_i(1)) \dots\dots\dots\dots (27)$$

$$y(x) = x + \sum_{i=1}^{N} a_i\, (Q_i(x) - xQ_i(1)) \dots\dots\dots\dots (28)$$

Substituting equations (23) and (28) in (21), leads to

$$\sum_{i=1}^{N} a_i(C_i(x) + Q_i(x) - xQ_i(1)) = 1 - x \dots\dots\dots\dots (29)$$

Taking the collocation points $x = x_j$ in equations (29) and (28), we have

$$\sum_{i=1}^{N} a_i\, (C_i(x_j) + Q_i(x_j) - x_j Q_i(1)) = 1 - x_j \dots\dots\dots\dots (30)$$

$$y(x_j) = x_j + \sum_{i=1}^{N} a_i\, (Q_i(x_j) - x_j Q_i(1)) \dots\dots\dots\dots (31)$$

The N system of equations in equation (30) are solved in order to determine the wavelet coefficients $a_i, i = 1,2, \dots, N$. Then the Chebyshev wavelet solutions at the collocation points $x_j, j = 1,2, \dots, N$ are obtained by using the wavelet coefficients values in (31). In

Figure **3** we plot the numerical solution and exact solution and values are tabulated in Table 3 for $k = 3, M = 10$.

EXAMPLE 4:

Consider the nonlinear ordinary differential equation with boundary conditions

$$y'' - yy' = -\frac{1}{4\sqrt{(1+x)^3}} - \frac{1}{2} \dots\dots\dots\dots (32)$$

$$y(0) = 1\,, y(1) = \sqrt{2} \dots\dots\dots\dots (33)$$

The exact solution is $y(x) = \sqrt{1+x}$.

As illustrated in example 3, we have

$$y''(x) = \sum_{i=1}^{N} a_i\, C_i(x) \dots\dots\dots\dots (34)$$

$$y'(x) = \sqrt{2} - 1 + \sum_{i=1}^{N} a_i \left(P_i(x) - Q_i(1)\right) \dots\dots\dots\dots (35)$$

$$y(x) = 1 + (\sqrt{2} - 1)x + \sum_{i=1}^{N} a_i \left(Q_i(x) - xQ_i(1)\right) \dots\dots\dots\dots (36)$$

where $a_i's, i = 1,2,\dots,N$ are Chebyshev wavelet coefficients to be determined. The nonlinear terms in equation (32) are handled using quasilinearization technique, which leads to

$$y''_{r+1} - y'_{r+1}\, y_r - y_{r+1} y'_r = -y_r\, y'_r - \frac{1}{4\sqrt{(1+x)^3}} - \frac{1}{2} \dots\dots\dots\dots (37)$$

Substituting equations (34) and (36) in equation (37) and taking the collocation points $x = x_j$, we obtain

$$\sum_{i=1}^{N} a_i\left(C_i(x_j) - y_r\left(P_i(x_j) - Q_i(1)\right) - y'_r\left(Q_i(x_j) - x_j Q_i(1)\right)\right) = \left(\sqrt{2} - 1\right)y_r$$
$$+\left(1 + \left(\sqrt{2} - 1\right)x_j\right)y'_r - y_r\, y'_r - \frac{1}{4\sqrt{\left(1 + x_j\right)^3}} - \frac{1}{2} \dots\dots\dots\dots (38)$$

$$y(x_j) = 1 + \left(\sqrt{2} - 1\right)x_j + \sum_{i=1}^{N} a_i \left(Q_i(x_j) - x_j Q_i(1)\right) \dots\dots\dots\dots (39)$$

Using equation (38), we obtain the wavelet coefficients $a_i, i = 1,2,\dots,N$ which are substituted in equation (39) to determine the Chebyshev wavelet solutions at the collocation points $x_j, j = 1,2,\dots,N$. In Table 4 we compare the results obtained by CWCM with the exact solutions for $k = 3, M = 10$ and the behaviour of the solution is shown in

Figure **4**.

4 CONCLUSION

In this paper, the Chebyshev wavelet collocation method is applied to linear and nonlinear, initial and boundary value problems. The method converts differential equation to system of algebraic equations which can be solved easily. The results obtained are in good agreement with the exact solution. The accuracy of the results are high even for small number of grid points. We analyse from the tables and graphs that the CWCM solutions are very close to the exact solutions. In the case of boundary value problems this method is very convenient as it takes care of the boundary conditions automatically. Thus the method is simple, reliable and highly efficient.

Table 1: Comparison of CWCM and exact solution of example 1.

x	y(x)	
	CWCM	**Exact**
0.125000	2.317301	2.317399
1.125000	16.215125	16.219907
2.125000	131.911217	131.957360
3.125000	1012.759350	1013.309691
4.125000	7588.276111	7593.399997
5.125000	56354.014350	56396.915645
6.125000	417090.821054	417505.435214
7.125000	3083782.397952	3087106.286808
8.125000	22787375.548094	22816605.660021
9.125000	168369534.307553	168608956.465621

Table 2: Comparison of CWCM and exact solution of example 2.

x	y(x)	
	CWCM	**Exact**
0.125000	0.9921976670	0.9921976672
1.125000	0.4311765159	0.4311765168
2.125000	-0.5262663355	-0.5262663347
3.125000	-0.9998623443	-0.9998623451
4.125000	-0.5541895254	-0.5541895265
5.125000	0.4010025879	0.4010025870
6.125000	0.9875147715	0.9875147713
7.125000	0.6661104290	0.6661104290
8.125000	-0.2677127712	-0.2677127697
9.125000	-0.9554020839	-0.9554020827

Table 3: Comparison of CWCM and exact solution of example 3.

x	y(x)	
	CWCM	**Exact**
0.012500	0.00810407266904134	0.00810407266904123

0.112500	0.07840459771933983	0.07840459771933983
0.212500	0.15791339939028937	0.15791339939028937
0.312500	0.24583605201770939	0.24583605201770942
0.412500	0.34129406151991160	0.34129406151991160
0.512500	0.44333364302017275	0.44333364302017264
0.612500	0.55093525074994343	0.55093525074994343
0.712500	0.66302376501315008	0.66302376501315008
0.812500	0.77847923442681177	0.77847923442681177
0.912500	0.89614806610505104	0.89614806610505093

Table 4: Comparison of CWCM and exact solution of example 4.

x	**y(x)**	
	CWCM	**Exact**
0.012500	1.006230589874905	1.006230589874905
0.112500	1.054751155486445	1.054751155486449
0.212500	1.101135777277254	1.101135777277262
0.312500	1.145643923738950	1.145643923738960
0.412500	1.188486432400462	1.188486432400471
0.512500	1.229837387624876	1.229837387624884
0.612500	1.269842509920022	1.269842509920029
0.712500	1.308625232830234	1.308625232830240
0.812500	1.346291201783623	1.346291201783626
0.912500	1.382931668593931	1.382931668593933

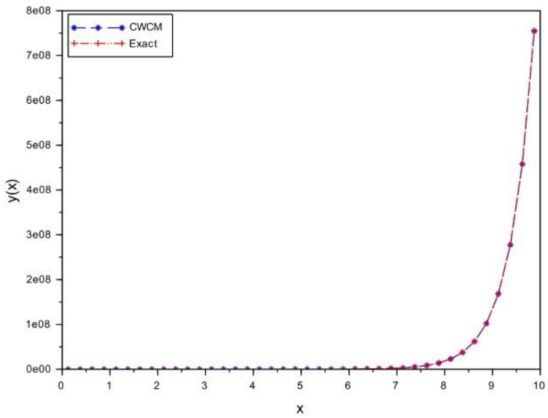

Figure 1: Comparison of CWCM and exact solution of example 1.

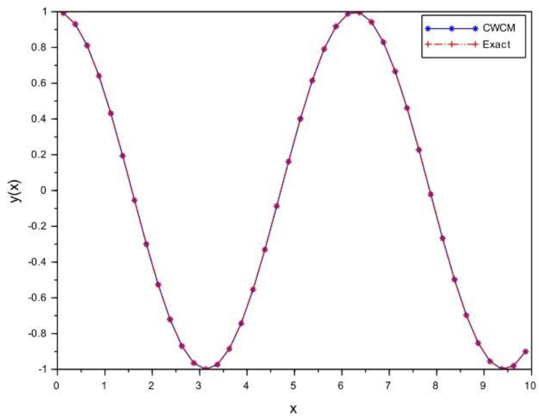

Figure 2: Comparison of CWCM and exact solution of example 2.

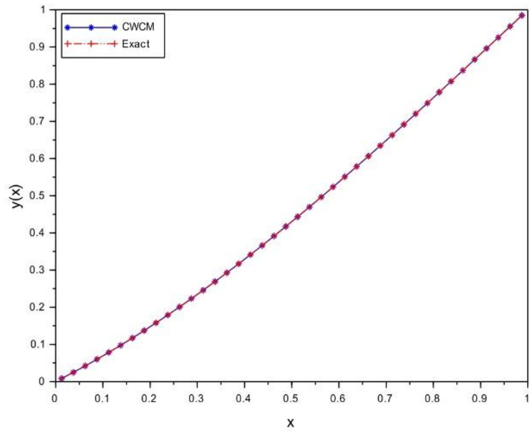

Figure 3: Comparison of CWCM and exact solution of example 3.

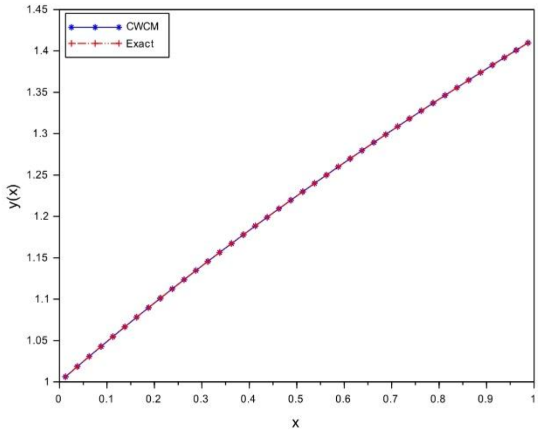

Figure 4: Comparison of CWCM and exact solution of example 4.

REFERENCES

1. Debnath, L., & Shah, F. A. (2002). Wavelet transforms and their applications (pp. 12-14). Boston: Birkhäuser.
2. Çelik, İ. (2018). Chebyshev Wavelet collocation method for solving a class of linear and nonlinear nonlocal boundary value problems. Fundamental Journal of Mathematics and Applications, 1(1), 25-35.
3. Rajini, G. K. (2016). A comprehensive review on wavelet transform and its applications. ARPN Journal of Engineering and Applied Sciences, 11(19), 11-713.
4. Chun-Lin, L. (2010). A tutorial of the wavelet transform. NTUEE, Taiwan, 21,

22.

5. Awashie, G. E., Amoako-Yirenkyi, P., & Dontwi, I. K. (2019). Chebyshev wavelets collocation method for simulating a two-phase flow of immiscible fluids in a reservoir with different capillary effects. Journal of Petroleum Exploration and Production Technology, 9(3), 2039-2051.
6. Adibi, H., & Assari, P. (2010). Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. Mathematical problems in Engineering, 2010.
7. Hosseini, S. G., & Mohammadi, F. (2011). A new operational matrix of derivative for Chebyshev wavelets and its applications in solving ordinary differential equations with non-analytic solution. Applied mathematical sciences, 5(51), 2537-2548.
8. Celik, I. (2013). Numerical solution of differential equations by using Chebyshev wavelet collocation method. Cankaya university journal of science and engineering, 10(2).
9. Shiralashetti, S. C., & Deshi, A. B. (2017). Chebyshev Wavelet Collocation Method for the Numerical Solution of Ordinary Differential Equations. Journal of the Nigerian Mathematical Society, 36(2), 337-353.
10. Heydari, M. H., Hooshmandasl, M. R., & Ghaini, F. M. (2014). A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type. Applied Mathematical Modelling, 38(5-6), 1597-1606.
11. Oruç, Ö. Bulut, F., & Esen, A. L. A. A. T. T. İ. N. (2019). Chebyshev wavelet method for numerical solutions of coupled Burgers' equation. Hacettepe Journal of Mathematics and Statistics, 48(1), 1-16.
12. Usman, M., Zubair, T., Hamid, M., Haq, R. U., & Wang, W. (2018). Wavelets solution of MHD 3-D fluid flow in the presence of slip and thermal radiation effects. Physics of Fluids, 30(2), 023104.
13. Bellman, R. E., & Quasilinearization, R. K. (1965). Nonlinear Boundary-Value Problems. American Elsevier Publishing Co., Inc., New York.
14. Saeed, U. (2014, January). Wavelet-Galerkin quasilinearization method for nonlinear boundary value problems. In Abstract and Applied Analysis (Vol. 2014). Hindawi.
15. Iqbal, M. A., Khan, U., Ali, A., & Mohyud-Din, S. T. (2015). Shifted Chebyshev wavelet-quasilinearization technique for MHD squeezing flow between two infinite plates and Jeffery–Hamel flows. Egyptian Journal of Basic and Applied Sciences, 2(3), 229-235.