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HYERS-ULAM STABILITY QUADRATIC β-FUNCTIONAL INEQUALITIES WITH THREE VARIABLES IN γ-HOMOGENEOUS NORMED SPACE

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Abstract

In this paper, we study to solve two quadratic β-functional inequalities with three variables in γ-homogeneous complex Banach spaces and prove the Hyers-Ulam stability of quadratic β-functional equations assosiated two the quadratic β-functional inequalities in γ-homogeneous complex Banach spaces. We will show that the solutions of the first and second inequalities are quadratic mappings.

Keywords: Hyers-Ulam stability γ-homogeneous space; quadratic β- functional equation; β- functional inequality

Mathemtics subject clssification (2010): 39B62, 39B72, Secondary 39B52, 39B8

1. INTRODUCTION

Let X and Y be a $\gamma-homogeneous$ normed spaces on the same filed $\mathbb{K}$, and $f : X \to Y$ be a mapping. We use the notation $\left\|\cdot\right\|$ for the norms on both X and Y. In this paper, we investigate first functional inequalities when X is a $\gamma - homogeneous$ real or complex Banach space and Y is a $\gamma - homogeneous$ complex Banach space

$$\begin{aligned} &\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f(z)\right\| \\ \leq\ &\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f(z)\right)\right\| \quad (1.1)\end{aligned}$$

where β is fixed complex number with $\left|\beta\right| < 1$, and

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$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\big(z\big)\right\|\\ \leq\ &\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\big(z\big)\right)\right\| \quad (1.2)\end{aligned}$$

where β is a fixed complex number with $\left|\beta\right| < \frac{1}{2}$
The notions of homogeneous real or complex Banach space will remind in the next section. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [24, 25] concerning the stability of group homomorphisms.
The functional equation

$$f\Big(x+y\Big)=f\Big(x\Big)+f\Big(y\Big)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The Hyers [9] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers'Theorem was generalized by Aoki[1] additive mappings and by Rassias [17] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f\Big(x\Big)+\frac{1}{2}f\Big(y\Big)$$

is called the $Jensen\ equation$.
The functional equations

$$f\Big(x+y\Big)+f\Big(x-y\Big)=2f\Big(x\Big)+2f\Big(y\Big)$$

is called the $quadratic$ functional $equations$.
The functional equations

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=\frac{1}{2}f\Big(x\Big)+\frac{1}{2}f\Big(y\Big)$$

is called the Jensen type $quadratic$ functional $equations$. See [11, 12, 13]
for more information on functional equations. The Hyers-Ulam stability for functional inequalities have been investigated such as in [7]. Gilany showed that is if satisfies the functional inequality

$$\left\|2f\Big(x\Big)+2f\Big(y\Big)-f\Big(xy^{-1}\Big)\right\|\leq\left\|f\Big(xy\Big)\right\| \quad (1.3)$$

Then f satisfies the Jordan-von Newman functional equation

$$2f\Big(x\Big) + 2f\Big(y\Big) = f\Big(xy\Big) + f\Big(xy^{-1}\Big)$$

See also [27]. Gil$\acute{a}$nyi [7,8] and Fechner [6] proved the Hyers-Ulam stability of the functional inequality.
Choonkil $Park^a$[31] proved the quadratic ρ-functional inequalities. Recently, in [18, 31, 32] the authors studied the Hyers-Ulam stability for the following quadraic functional inequalities

$$\begin{aligned}&\left\|f\Big(x+y\Big) \ + f\Big(x+y\Big) - 2f\Big(x\Big) - 2f(y)\right\|\\ &\qquad \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f\Big(x\Big) - f\Big(y\Big)\right)\right\|\end{aligned} \tag{1.4}$$

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right) \ + 2f\left(\frac{x-y}{2}\right) - f\Big(x\Big) - f\Big(y\Big)\right\|\\ &\qquad \leq \left\|\rho\left(\left(f\Big(x+y\Big) + f\Big(x+y\Big) - 2f\Big(x\Big) - 2f\Big(y\Big)\right)\right\|\end{aligned} \tag{1.5}$$

in homogeneous real (complex) Banach spaces

In this paper, we solve and proved the Hyers-Ulam stability for two quadratic β-functional inequalities (1.1)-(1.2), ie the quadratic β -functional inequalities with three variables [11, 13, 14, 18]. Under suitable assumptions on spaces X and Y, we will prove that the mappings satisfying the quadratic β-functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [31] for quadratic β-functional inequatilies with three variables.
The paper is organized as followns: In section preliminarier we remind some basic notations in [28] such as $F - norm$ is called γ-homogeneous $\Big(\gamma > 0\Big)$.

Section 3 is devoted to prove the Hyers-Ulam stability of the quadratic β- functional inequalities($\big|\beta\big| < 1$) (1.1) and (1.2) when X is γ_1-homogeneous $\Big(\gamma_1 \leq 1\Big)$ real or complex normed space and Y is γ_2-homogeneous $\Big(\gamma_2 \leq 1\Big)$ complex Bananch space.

Section 4 is devoted to prove the Hyers-Ulam stability of the quadratic β- functional inequalities $\Big(\Big|\beta\Big| < \frac{1}{2}\Big)$ (1.1) and (1.2) when X is γ_1-homogeneous $\Big(\gamma_1 \leq 1\ \Big)$ real or complex normed space and Y is γ_2-homogeneous $\Big(\gamma_2 \leq 1\Big)$ complex Bananch space.

2. PRELIMINARIER

2.1 F^*- spaces.

Definition 2.1. Let X be a linear space. A nonnegative valued function $\Big\|\cdot\Big\|$ is an $F-norm$ if it satisfies the following conditions:

1. $\Big\|x\Big\| = 0$ if and only if $x = 0$;
2. $\Big\|\lambda x\Big\| = \Big\|x\Big\|$ for all $x \in X$ and all λ with $\Big|\lambda\Big| = 1$;
3. $\Big\|x + y\Big\| \leq \Big\|x\Big\| + \Big\|y\Big\|$ for all $x, y \in X$;
4. $\Big\|\lambda_n x\Big\| \to 0$, $\lambda_n \to 0$;
5. $\Big\|\lambda_n x\Big\| \to 0$, $x_n \to 0$.

Then $\Bigg(X, \Big\|\cdot\Big\|\Bigg)$ is called an F^*-space. An F-space is a complete F^*-space. An F-norm is called β-homgeneous ($\beta > 0$) if $\Big\|tx\Big\| = \Big|t\Big|^\beta\Big\|x\Big\|$ for all $x \in X$ and $t \in \mathbb{C}$.

2.2 Solutions of the inequalities.

The functional equation

$$f\Big(x + y\Big) = f\Big(x\Big) + f\Big(y\Big)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an $additive\ mapping$.
The functional equation $f\Big(\frac{x+y}{2}\Big) = \frac{1}{2}f\Big(x\Big) + \frac{1}{2}f\Big(y\Big)$ called the $Jensen\ equation$. The functional equation

$$f\Big(x + y\Big) + f\Big(x - y\Big) = 2f\Big(x\Big) + 2f\Big(y\Big)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [27] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewe [5] noticed that the theorm of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.
The functional equation

$$2\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f\left(x\right)+f\left(y\right)$$

is called a $Jensen\ type\ quadratic\ equation.$ See [11, 12, 13]

3. QUADRATIC Γ-FUNCTIONAL INEQUALITY

In This section, assume that β is a fixed complex number with $\left|\beta\right| < 1$. We investigate the quadratic β-fuctional inequality (1.1) in γ-homogeneous complex Banach space.

Lemma 3.1. *A mapping $f : X \to Y$ satilies*

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right\|\\ \leq\ &\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\| \qquad (3.1)\end{aligned}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadraic.

Proof. Assume that $f : X \to$Y satisfies (3.1)
Letting $x = y = z = 0$ in (3.1), we get
$\left\|2f(0)\right\| \leq \left|\beta\right|^{\gamma_2}\left\|2f(0)\right\|$.
So $f(0) = 0$
Letting x = y = z in (3.1), we get
$\left\|f\left(2x\right)-4f\left(x\right)\right\| \leq 0$ and so f$\left(2\text{x}\right)$ = 4f$\left(\text{x}\right)$ for all $x \in$X.
Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{4}f\left(x\right) \qquad (3.2)$$

for all $x \in X$

It follows from (3.1) and (3.2) that:

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right\|\\ \leq\ &\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\|\\ =\ &\frac{\left|\beta\right|^{\gamma_2}}{2^{\gamma_2}}\left\|\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}+z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \end{aligned} \tag{3.3}$$

and so

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)=2f\left(\frac{x+y}{2}\right)+2f(z)$$

. for all $x, y, z \in X$

The coverse is obviously true. □

Lemma 3.2. *A mapping $f : X \to Y$ satilies*

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\|\\ \leq\ &\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \end{aligned} \tag{3.4}$$

for all x,y,z∈ X if and only if f:X→Y is quadratic

Proof. Assume that $f : X \to$Y satisfies (3.4)

Letting $x = y = z = 0$ in (3.4), we get

$\left\|2f(0)\right\| \leq \left|\beta\right|^{\gamma_2}\left\|2f\left(0\right)\right\|$.

So $f(0) = 0$

Letting $x = y$; $z = 0$ in (3.4), we get

$\left\|4f\left(\frac{x}{2}\right)-f\left(x\right)\right)\right\| \leq 0$ and so $4f\left(\frac{x}{2}\right)=f\left(x\right)$ for all x ∈X. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{4}f\left(x\right) \tag{3.5}$$

for all $x \in X$

It follows from (3.4) and (3.5) that:

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\| \\
=\ & \frac{1}{2^{\gamma_2}}\left\|\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \\
\leq\ & \left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \\
=\ & \left|\beta\right|^{\gamma_2}\left\|\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\| \qquad (3.6)
\end{aligned}$$

and so

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)=2f\left(\frac{x+y}{2}\right)+2f\left(z\right).$$

for all $x, y, z \in X$

The coverse is obviously true. □

Theorem 3.3. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned}
&\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right\| \\
&\qquad\leq\left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\| \\
&\qquad+\theta\left(\left\|x\right\|^r+\left\|y\right\|^r+\left\|z\right\|^r\right) \qquad (3.7)
\end{aligned}$$

for all x,y,z$\in X$. Then there exists a unique quadratic mapping $h : X \to$Y such that

$$\left\|f\left(x\right)-h\left(x\right)\right\| \leq \frac{3\theta}{2^{\gamma_1 r}-4^{\gamma_2}}\left\|x\right\|^r \qquad (3.8)$$

for all $x \in X$

Proof. Letting $x = y = z = 0$ in (3.7), we get $\left\|2f\left(0\right)\right\| \leq \left|\beta\right|^{\gamma_2}\left\|2f\left(0\right)\right\|$. So $f\left(0\right) = 0$. Letting $x = y = z$ in (3.7), we get

$$\left\|f\left(2x\right)-4f\left(x\right)\right\| \leq 3\theta\left\|x\right\|^r. \qquad (3.9)$$

for all $x \in X$. So

$$\left\|f\left(x\right) - 4f\left(\frac{x}{2}\right)\right\| \le \frac{3}{2^{\gamma_1 r}}\theta\left\|x\right\|^r$$

So for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{3}{2^{\gamma_1 r}}\sum_{j=l}^{m-1}\frac{4^{\gamma_2 j}}{2^{\gamma_1 r j}} \end{aligned} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{4^n f(\frac{x}{2^n})\right\}$ coverges. So one can define te mapping $h : X \to Y$ by

$$h\left(x\right) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.8) It follows from (3.7) that

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) - 2f\left(\frac{x+y}{2}\right) - 2f\left(z\right)\right\| \\ &= \lim_{n\to\infty} 4^{\gamma_2 n}\left\|f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right) + f\left(\frac{x+y}{2^{n+1}}-\frac{z}{2^n}\right) - 2f\left(\frac{x+y}{2^{n+1}}\right) - 2f\left(\frac{z}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 4^{\gamma_2 n}\left|\beta\right|^{\gamma_2}\left\|2f\left(\frac{x+y}{2^{n+2}}+\frac{z}{2^{n+1}}\right) + 2f\left(\frac{x+y}{2^{n+2}}-\frac{z}{2^{n+1}}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\ &+ \lim_{n\to\infty}\frac{4^{\gamma_2 n}\theta}{2^{\gamma_1 n r}}\left(\left\|x\right\|^r + \left\|y\right\|^r + \left\|z\right\|^r\right) \\ &= \left|\beta\right|^{\gamma_2}\left\|2h\left(\frac{x+y}{2}+z\right) + 2h\left(\frac{x+y}{2}-z\right) - h\left(\frac{x+y}{2}\right) - h\left(z\right)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x+y}{2}-z\right) - 2f\left(\frac{x+y}{2}\right) - 2f\left(z\right)\right\| \\ &\qquad \le \left\|\beta\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right) + 2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(z\right)\right)\right\|\end{aligned}$$

for all $x, y \in X$. By $Lemma$ 3.1, the mapping $h : X \to Y$ is quadratic. Now, let $T : X \to Y$ be another quadratic mapping sattisfying (3.8). Then we have

$$\begin{aligned}\left\|h\left(x\right) - T\left(x\right)\right\| &= 4^{\gamma_2 n}\left\|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\leq 4^{\gamma_2 n}\left(\left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\leq \frac{6.4^{\gamma_2 n}}{\left(2^{\gamma_1 r} - 4^{\gamma_2}\right)2^{\gamma_1 n r}}\theta\left\|x\right\|\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h\left(x\right) = T\left(x\right)$ for all $x \in X$. This prover the uniqueness of h. Thus the mapping $h : X \to Y$ be a unique mapping satisfiying (3.8). □

Theorem 3.4. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned}&\left\|f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f\left(z\right)\right\| \\ \leq\ &\left\|\beta\left(2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + 2f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(z\right)\right)\right\| \\ +\ &\theta\left(\left\|x\right\|^r + \left\|y\right\|^r + \left\|z\right\|^r\right) \qquad (3.11)\end{aligned}$$

for all x,y,z$\in X$. Then there exists a unique quadratic mapping $h : X \to$Y such that

$$\left\|f\left(x\right) - h\left(x\right)\right\| \leq \frac{3\theta}{4^{\gamma_2} - 2^{\gamma_1 r}}\left\|x\right\|^r \qquad (3.12)$$

for all $x \in X$

Proof. Letting $x = y = z = 0$ in (3.11), we get $\left\|2f\left(0\right)\right\| \leq \left|\beta\right|^{\gamma_2}\left\|2f\left(0\right)\right\|$. So $f\left(0\right) = 0$. Letting $x = y = z$ in (3.11), we get

$$\left\|f\left(2x\right) - 4f\left(x\right)\right\| \leq 3\theta\left\|x\right\|^r \qquad (3.13)$$

. for all $x \in X$.So

$$\left\|f\left(x\right) - \frac{1}{4}f\left(2x\right)\right\| \leq \frac{3\theta}{4^{\gamma_2}}\left\|x\right\|^r$$

So for all $x \in X$. Hence

$$\begin{aligned}\Big\|\frac{1}{4^l}f\Big(2^lx\Big) & -\frac{1}{4^m}f\Big(2^mx\Big)\Big\| \\ & \leq \sum_{j=l}^{m-1}\Big\|\frac{1}{4^j}f\Big(2^jx\Big)-\frac{1}{4^{j+1}}f\Big(2^{j+1}x\Big)\Big\| \\ & \leq \frac{3\theta}{4^{\gamma_2}}\sum_{j=l}^{m-1}\frac{2^{\gamma_1 rj}}{4^{\gamma_2 j}}\Big\|x\Big\|^r \end{aligned} \tag{3.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.14) that the sequence $\Big\{\frac{1}{4^n}f(2^nx)\Big\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\Big\{\frac{1}{4^n}f(2^nx)\Big\}$ coverges. So one can define the mapping $h : X \to Y$ by

$$h\Big(x\Big) := \lim_{n\to\infty}\frac{1}{4^n}f\Big(2^nx\Big)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.14), we get (3.12) It follows from (3.11) that

$$\begin{aligned}&\Big\|h\Big(\frac{x+y}{2}+z\Big)+h\Big(\frac{x+y}{2}-z\Big)-2h\Big(\frac{x+y}{2}\Big)-2h\Big(z\Big)\Big\| \\ &= \lim_{n\to\infty}\frac{1}{4^{\gamma_2 n}}\Big\|f\Big(2^{n-1}(x+y)+2^nz\Big)+f\Big(2^{n-2}\Big(x+y\Big)-2^{n-1}z\Big) \\ &-2f\Big(2^{n-1}(x+y)\Big)-2f\Big(2^nx\Big)\Big\| \\ &\leq \lim_{n\to\infty}\frac{1}{4^{\gamma_2 n}}\Big|\beta\Big|^{\gamma_2}\Big\|\Big(2f\Big(2^{n-2}(x+y)+2^{n-1}z\Big)+2f\Big(2^{n-2}(x+y)-2^{n-1}z\Big) \\ &-f\Big(2^{n-1}\Big(x+y\Big)\Big)-f\Big(2^{n-1}\Big)\Big)\Big\|+\lim_{n\to\infty}\frac{2^{\gamma_1 n}\theta}{4^{\gamma_2 nr}}\Big(\Big\|x\Big\|^r+\Big\|y\Big\|^r+\Big\|z\Big\|^r\Big) \\ &= \Big|\beta\Big|^{\gamma_2}\Big\|\Big(2h\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2h\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-h\Big(\frac{x+y}{2}\Big)-h\Big(z\Big)\Big)\Big\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\Big\|f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+y}{2}-z\Big)-2f\Big(\frac{x+y}{2}\Big)-2f\Big(z\Big)\Big\| \\ &\qquad\leq \Big\|\beta\Big(2f\Big(\frac{x+y}{2^2}+\frac{z}{2}\Big)+2f\Big(\frac{x+y}{2^2}-\frac{z}{2}\Big)-f\Big(\frac{x+y}{2}\Big)-f\Big(z\Big)\Big)\Big\|\end{aligned}$$

for all $x, y \in X$. By $Lemma$ 3.1, the mapping $h : X \to Y$ is quadratic. Now, let $T : X \to Y$ be another quadratic mapping sattisfying (3.12). Then we have

$$\begin{aligned}\left\|h\left(x\right)-T\left(x\right)\right\| &= \frac{1}{4^{\gamma_2 n}}\left\|h\left(2^n x\right)-T\left(2^n x\right)\right\| \\ &\le \frac{1}{4^{\gamma_2 n}}\left(\left\|h\left(2^n x\right)-f\left(2^n x\right)\right\|+\left\|h\left(2^n x\right)-f\left(2^n x\right)\right\|\right) \\ &\le \frac{6.2^{\gamma_1 n}}{\left(4^{\gamma_2 r}-2^{\gamma_1}\right)4^{\gamma_2 n r}}\theta.\left\|x\right\|^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This prover the uniqueness of h. Thus the mapping $h : X \to Y$ be a unique mapping satisfiying (3.12). □

4. QUADRATIC γ-FUNCTIONAL INEQUALITY

In This section, assume that β is a fixed complex number with $\left|\beta\right| < \frac{1}{2}$. We investigate the quadratic β-fuctional inequality (1.2) in γ-homogeneous complex Banach space.

Theorem 4.1. *Let $r > \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\| \\ &\qquad \le \left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\| \\ &\qquad + \theta\left(\left\|x\right\|^r+\left\|y\right\|^r+\left\|z\right\|^r\right) \qquad (4.1)\end{aligned}$$

for all x,y,z$\in X$. Then there exists a unique quadratic mapping $h : X \to$Y such that

$$\left\|f\left(x\right)-h\left(x\right)\right\| \le \frac{2^{\gamma_1 r}\theta}{2^{\gamma_1 r}-4^{\gamma_2}}\left\|x\right\|^r \qquad (4.2)$$

for all $x \in X$

Proof. Letting $x = y = z = 0$ in (4.1), we get $\left\|2f\left(0\right)\right\| \le \left|\beta\right|^{\gamma_2}\left\|2f\left(0\right)\right\|$. So $f\left(0\right) = 0$. Letting $x = y; z = 0$ in (4.1), we get

$$\left\|4f\left(\frac{x}{2}\right)-f\left(x\right)\right\| \le 2\theta\left\|x\right\|^r. \qquad (4.3)$$

So for all $x \in X$. Hence

$$\begin{aligned}\Big\|4^l f\Big(\frac{x}{2^l}\Big) &- 4^m f\Big(\frac{x}{2^m}\Big)\Big\| \\ &\leq \sum_{j=l}^{m-1}\Big\|4^j f\Big(\frac{x}{2^j}\Big) - 4^{j+1} f\Big(\frac{x}{2^{j+1}}\Big)\Big\| \\ &\leq 2\sum_{j=l}^{m-1}\frac{4^{\gamma_2 j}}{2^{\gamma_1 r j}}\Big\|x\Big\|^r \end{aligned} \tag{4.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.4) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ coverges. So one can define te mapping $h : X \to Y$ by

$$h\Big(x\Big) := \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.4), we get (4.2) It follows from (4.1) that

$$\begin{aligned}\Big\|2h\Big(\frac{x+y}{2^2} &+ \frac{z}{2}\Big) + 2h\Big(\frac{x+y}{2^2} - \frac{z}{2}\Big) - h\Big(\frac{x+y}{2}\Big) - h\Big(z\Big)\Big\| \\ &= \lim_{n\to\infty} 4^{\gamma_2 n}\Big\|2f\Big(\frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}}\Big) + 2f\Big(\frac{x+y}{2^{n+2}} - \frac{z}{2^{n+1}}\Big) - f\Big(\frac{x+y}{2^{n+1}}\Big) - f\Big(\frac{z}{2^n}\Big)\Big\| \\ &\leq \lim_{n\to\infty} 4^{\gamma_2 n}\Big|\beta\Big|^{\gamma_2}\Big\|\Big(f\Big(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\Big) + f\Big(\frac{x+y}{2^{n+1}} - \frac{z}{2^n}\Big) - 2f\Big(\frac{x+y}{2^{n+1}}\Big) - 2f\Big(\frac{z}{2^n}\Big)\Big)\Big\| \\ &+ \lim_{n\to\infty}\frac{4^{\gamma_2 n}\theta}{2^{\gamma_1 n r}}\Big(\Big\|x\Big\|^r + \Big\|y\Big\|^r + \Big\|z\Big\|^r\Big) \\ &= \Big|\beta\Big|^{\gamma_2}\Big\|h\Big(\frac{x+y}{2} + z\Big) + h\Big(\frac{x+y}{2} - z\Big) - 2h\Big(\frac{x+y}{2}\Big) - 2h\Big(z\Big)\Big\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}\Big\|2f\Big(\frac{x+y}{2^2} + \frac{z}{2}\Big) &+ 2f\Big(\frac{x+y}{2^2} - \frac{z}{2}\Big) - f\Big(\frac{x+y}{2}\Big) - f\Big(z\Big)\Big\| \\ &\leq \Big\|\beta\Big(f\Big(\frac{x+y}{2} + z\Big) + f\Big(\frac{x+y}{2} - z\Big) - 2f\Big(\frac{x+y}{2}\Big) - 2f\Big(z\Big)\Big)\Big\|\end{aligned}$$

for all $x, y \in X$. By *Lemma* 3.2, the mapping $h : X \to Y$ is quadratic. Now, let

$T : X \to Y$ be another quadratic mapping sattisfying (4.2). Then we have

$$\begin{aligned}\left\|h\Big(x\Big) - T\Big(x\Big)\right\| &= 4^{\gamma_2 n}\left\|h\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\right\| \\ &\leq 4^{\gamma_2 n}\left(\left\|h\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big)\right\| + \left\|h\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big)\right\|\right) \\ &\leq \frac{2\cdot\theta}{\left(2^{\gamma_1 r} - 4^{\beta_2}\right)2^{\gamma_1 r}}\cdot\frac{4^{\gamma_2 n}}{2^{\gamma_1 nr}}\left\|x\right\|^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This prover the uniqueness of h. Thus the mapping $h : X \to Y$ be a unique mapping satisfiying (4.2). □

Theorem 4.2. *Let $r < \frac{2\gamma_2}{\gamma_1}$ and θ be positive real numbers, and let $f : X \to Y$ is a mapping satisfying*

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + 2f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - f\left(\frac{x+y}{2}\right) - f\Big(z\Big)\right\| \\ &\qquad\leq \left\|\beta\left(f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f\Big(z\Big)\right)\right\| \\ &\qquad + \theta\left(\left\|x\right\|^r + \left\|y\right\|^r + \left\|z\right\|^r\right) \qquad (4.5)\end{aligned}$$

for all x,y,z$\in X$. Then there exists a unique quadratic mapping $h : X \to$Y such that

$$\left\|f\Big(x\Big) - h\Big(x\Big)\right\| \leq \frac{2^{\gamma_1 r}\theta}{4^{\gamma_2} - 2^{\gamma_1 r}}\left\|x\right\|^r \qquad (4.6)$$

for all $x \in X$

Proof. Letting $x = y = z = 0$ in (4.5), we get $\left\|2f\Big(0\Big)\right\| \leq \left|\beta\right|^{\gamma_2}\left\|2f\Big(0\Big)\right\|$. So $f\Big(0\Big) = 0$. Letting $x = y; z = 0$ in (4.5), we get

$$\left\|4f\Big(\frac{x}{2}\Big) - f\Big(x\Big)\right\| \leq 2\theta\left\|x\right\|^r \qquad (4.7)$$

. So for all $x \in X$. for all $x \in X$.So

$$\left\|f\Big(x\Big) - \frac{1}{4}f\Big(2x\Big)\right\| \leq \frac{2^{(\gamma_1+1)r}\theta}{4^{\gamma_2}}\left\|x\right\|^r$$

Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f\Big(2^lx\Big) \ - \frac{1}{4^m}f\Big(2^mx\Big)\right\| \\
&\qquad\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\Big(2^jx\Big) - \frac{1}{4^{j+1}}f\Big(2^{j+1}x\Big)\right\| \\
&\qquad\leq \sum_{j=l}^{m-1}\frac{2^{(\gamma_1+1)jr}}{4^{\gamma_2 j}}\Big\|x\Big\|^r
\end{aligned} \tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\left\{\frac{1}{4^n}f\Big(2^nx\Big)\right\}$ is Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{4^n}f\Big(2^nx\Big)\right\}$ coverges. So one can define te mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.6) It follows from (4.5) that

$$\begin{aligned}
&\left\|2h\left(\frac{x+y}{2^2} \ + \frac{z}{2}\right) + 2h\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - h\left(\frac{x+y}{2}\right) - h\Big(z\Big)\right\| \\
&\qquad = \lim_{n\to\infty}\frac{1}{4^{\gamma_2 n}}\left\|2f\left(2^{n-2}(x+y) + 2^{n-1}z\right) + 2f\left(2^{n-2}(x+y) - 2^{n-1}z\right)\right. \\
&\qquad \left. - f\left(2^{n-1}(x+y)\right) - f\Big(2^nz\Big)\right\| \\
&\qquad \leq \lim_{n\to\infty}\frac{1}{4^{\gamma_2 n}}\Big|\beta\Big|^{\gamma_2}\left\|f\left(2^{n-1}(x+y) + 2^nz\right) + f\left(2^{n+1}(x+y) - 2^nz\right)\right. \\
&\qquad \left. - 2f\left(2^{n-1}x+y\right) - 2f\Big(2^nz\Big)\right\| \\
&\qquad + \lim_{n\to\infty}\frac{2^{\gamma_1 nr}\theta}{4^{\gamma_2 n}}\left(\Big\|x\Big\|^r + \Big\|y\Big\|^r + \Big\|z\Big\|^r\right) \\
&\qquad = \Big|\beta\Big|^{\gamma_2}\left\|h\left(\frac{x+y}{2} + z\right) + h\left(\frac{x+y}{2} - z\right) - 2h\left(\frac{x+y}{2}\right) - 2h\Big(z\Big)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)+2f\left(\frac{x+y}{2^2}-\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\|$$
$$\leq\left\|\beta\left(f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+y}{2}-z\right)-2f\left(\frac{x+y}{2}\right)-2f\left(z\right)\right)\right\|$$

for all $x, y \in X$. By $Lemma$ 3.2, the mapping $h : X \to Y$ is quadratic. Now, let $T : X \to Y$ be another quadratic mapping sattisfying (4.6). Then we have

$$\begin{aligned}\left\|h\left(x\right)-T\left(x\right)\right\| &= \frac{1}{4^{\gamma_2 n}}\left\|h\left(2^n x\right)-T\left(2^n x\right)\right\| \\ &\leq \frac{1}{4^{\gamma_2 n}}\left(\left\|h\left(2^n x\right)-f\left(2^n x\right)\right\|+\left\|h\left(2^n x\right)-f\left(2^n x\right)\right\|\right) \\ &\leq \frac{2\cdot\theta}{\left(4^{\gamma_2}-2^{\gamma_1 r}\right)2^{\gamma_1 r}}\cdot\frac{2^{\gamma_1 n r}}{4^{\gamma_2 n}}\left\|x\right\|^r\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This prover the uniqueness of h. Thus the mapping $h : X \to Y$ be a unique mapping satisfiying (4.6). □

Remak 5.11
If β is a real number such that $-1 < \beta < 1$ and is Y is a γ_2-homogeneous real Banach space, then all the assertions in this sections remain valid

5. CONCLUSION

In this paper, I have shown that the solutions of the first and second quadratic β-functional inequalities are quadratic mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [18, 31, 32] .

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