International Journal of Mathematics Research.
ISSN 0976-5840 Volume 11, Number 1 (2019), pp. 43-54

http://www.irphouse.com

Coefficients Estimates for a Class of Q-Spirallike Function in the Space of Sigmoid Function

READ. S. A. QAHTAN[1], HAMID SHAMSAN[2], and S. LATHA[3]

[1,2,3]*Department of Mathematics, Yuvaraja's College, University of Mysore, Mysore 570005, India.*

Abstract

In the present we used subordination concept, and define some new subclasses of λ-q-spirallike functions and obtain the coefficients of λ-q-spirallike functions related to sigmoid functions and the Fekete-Szegö coefficients functional $|a_3 - \mu a_2^2|$ for certain normalized analytic functions definedon the open unit disk.

Keywords: Analytic functions, Subordination functions, q-spirallike functions, Sigmoid function, Fekete-Szegö Inequality.

AMS Subject Classification: 30C45

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \quad z^n, \tag{1}$$

defined on the unit disk $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ normalized by $f(0) = 0$, $f'(0) = 1$. Let S denote the subclass of function in $\mathcal{A}$ which are univalent in E.Special function is an information process that is inspired by the way nervous system such as the brain processes information. It composed of large number of highly interconnected processing elements (neurons) working together to solve a specific task. The sigmoid functions are extensively used in back propagation neural networks because it reduces the burden of complication involved during training phase. The theory of special functions has been developed by C. F.Gauss, C. G. J. Jacobi, F. Klein and many others in nineteenth century. However, in the twentieth century the theory of special functions was surpassing by other fields like real analysis, functional analysis, topology, algebra, differential equations and so on and nowadays is the theory of a special function. Example of special function is activation function. Activation function acts as a squashing function, such that the output of a neuron in a neural network is between certain values (Usually 0 and 1, or -1 and 1). There are three types of activation function, namely: threshold function, piecewise linear function and sigmoid function. The most popular activation function in the hardware implementation of artificial neural networks is the sigmoid function. Sigmoid function is often used with gradient descendent type learning algorithms. There are different possibilities for evaluating this function, such as truncated series expansion, look-up tables, or piecewise approximation. The sigmoid function is useful because it is differentiable, which is important for the weight-learning algorithms. The sigmoid function will increase the size of the hypothesis space that the network can represent. Neural networks can be used for complex learning tasks. The sigmoid function is of the form

$$h(z) = \frac{1}{1 + e^{-z}}. \tag{2}$$

Spacek [12] introduced the concept of spirallikeness which is a natural generalization of starlikeness. A function f in $\mathcal{A}$ is λ-spirallike if and only if,

$$\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in E, \tag{3}$$

where $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$.

Let the functions $f(z)$ and $g(z)$ be analytic in E Then we say that the function $f(z)$ is subordinate to $g(z)$ if there exits a Schwarz function $\omega(z)$, analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in E)$ such that $f(z) = g(\omega(z)), z \in E$. We denote this subordination by

$$f \prec g \ \ or \ \ f(z) \prec g(z).$$

Lemma 1.1. *[10] If a function $p \in P$is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in E),$$

then $|p_k| \leq 1, k \in N$ where P is the family of all functions analytic in E for $p(0) = 1$ and $\Re(p(z)) > 0,\ (z \in E)$.

In virtue of Lowners method, Fekete and Szego [2] proved the striking result, if $f \in S$ then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - \mu, & \text{if } \mu \leq 0, \\ 1 + 2e^{\frac{-2\mu}{1-\mu}}, & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \tag{4}$$

Let φ be an analytic function with positive real part in E such that $\varphi(0) = 1, \partial_q\varphi(0) > 0$ and $\varphi(E)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \tag{5}$$

where all coefficients are real and $B_1 > 0$. Recently, Oladipo[6], Murugusundaramoorthy et al [5], Olatunji et al [9] have studied sigmoid function for various classes of analytic and univalent functions. In [3] Jackson introduced and studied the concept of the q-derivative operator ∂_q as follows :

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z\,(1-q)}, \ \ (z \neq 0, \quad 0 < q < 1, \quad \partial_q f(0) = f'(0)). \tag{6}$$

Equivalently (6) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} \ [n]_q \ a_n \ z^{n-1}, \ \ z \neq 0, \tag{7}$$

where $[n]_q = \frac{1-\ q^n}{1-\ q}$, note that as $q \to 1^-$, $[n]_q \to n$.

Lemma 1.2. *[1] Let h be the sigmoid function defined in (2) and*

$$\Phi_{n,m}(z) = 2h(z) = 1 + \Sigma_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n} \ \ z^n \right)^m, \tag{8}$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.3. *[1] Let*

$$\Phi_{n,m}(z) = 1 + \Sigma_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n} \ z^n \right)^m, \tag{9}$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 1.4. *[1] If* $\Phi(z) \in P$ *and it is starlike, then* f *is a normalized univalent function of the form* (1)*. Taking* $m = 1$*, Joseph et al [1] remarked the following :*

Remark 1.1. *Let*

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} \ C_n \ z^n$$

.

where $C_n = \frac{(-1)^{n+1}}{2n}$ then $|c_n| \leq 2, n = 1, 2, 3, ...$ this result is sharp for each n.

Definition 1.1. *For* $b \in \mathbb{C} \setminus 0$*. Let the class* $\mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$ *denote the subclass of* $\mathcal{A}$ *consisting of functions f of the form* (1)*, and satisfying the following subordination condition*

$$1 + \frac{1}{b}\left[(1 + \tan\beta)\left(\frac{z\partial_q f(z) + \lambda z^2 \partial_q^2 f(z)}{\lambda z \partial_q f(z) + (1-\lambda) f(z)} - 1\right)\right] \prec \varphi(z). \tag{10}$$

or

$$1 + \frac{1}{b}\left[(1 + \tan\beta)\left(\frac{z\partial_q f(z) + \lambda z^2 \partial_q^2 f(z)}{\lambda z \partial_q f(z) + (1-\lambda) f(z)} - 1\right)\right] = \varphi(\omega(z)). \tag{11}$$

for $0 \leq \lambda \leq 1, \beta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \omega(z) = \frac{\Phi(z)-1}{\Phi(z)+1}$*and* $\Phi_{n,m}(z)$ *is a simple logistic sigmoid activation function.*

Definition 1.2. *For* $b \in \mathbb{C} \setminus 0$*. Let the class* $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m}, q)$ *denote the subclass of* $\mathcal{A}$ *consisting of functions f of the form* (1)*, and satisfying the following subordination condition*

$$1 + \frac{1}{b}\left[(1 + \tan\beta)\left(\frac{z\partial_q f(z)}{f(z)} + \lambda \frac{z^2 \partial_q^2 f(z)}{f(z)} - 1\right)\right] \prec \varphi(z). \tag{12}$$

or

$$1 + \frac{1}{b}\left[(1 + \tan\beta)\left(\frac{z\partial_q f(z)}{f(z)} + \lambda \frac{z^2 \partial_q^2 f(z)}{f(z)} - 1\right)\right] = \varphi(\omega(z)). \tag{13}$$

for $0 \leq \lambda \leq 1, \omega(z) = \frac{\Phi(z)-1}{\Phi(z)+1}$*, and* $\Phi_{n,m}(z)$ *is a simple logistic sigmoid activation function*

2. MAIN RESULT

Theorem 2.1. *If* $f \in \mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$ *given by* (10)*, then*

$$|a_2| = \frac{|b|B_1}{4\left[\left(([2]_q - 1) + \lambda\right)\right](1 + i\tan\beta)},$$

$$|a_3| = \frac{|b^2|B_1^2\,(1 + ([2]_q - 1)\lambda)}{16\left[([2]_q - 1) + \lambda\right]\left[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\lambda\right](1 + i\tan\beta)^2} + \frac{|b|(B_2 - B_1)}{16\left[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\lambda\right](1 + i\tan\beta)}.$$

Proof. If $f \in \mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$, then

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\left(\frac{z\partial_q f(z) + \lambda z^2\partial_q^2 f(z)}{\lambda z\partial_q f(z) + (1-\lambda)f(z)} - 1\right)\right] = \varphi(\omega(z)). \tag{14}$$

Define the function $\Phi(z)$ by

$$\Phi(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \ldots. \tag{15}$$

Or, equivalently

$$\omega(z) = \frac{\Phi(z) - 1}{\Phi(z) + 1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \ldots. \tag{16}$$

In view of (14) and (16), clearly we have

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\left(\frac{z\partial_q f(z) + \lambda z^2\partial_q^2 f(z)}{\lambda z\partial_q f(z) + (1-\lambda)f(z)} - 1\right)\right] = \varphi\left(\frac{\Phi(z) - 1}{\Phi(z) + 1}\right). \tag{17}$$

From the equations (5) and (16) , we have

$$\varphi\left(\frac{\Phi(z) - 1}{\Phi(z) + 1}\right) = 1 + \frac{B_1}{4}z + \frac{B_2 - B_1}{16}z^2 + \frac{3B_3 - 6B_2 - B_1}{192}z^3 + \ldots. \tag{18}$$

By simple calculations, we get

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\left(\frac{z\partial_q f(z) + \lambda z^2\partial_q^2 f(z)}{\lambda z\partial_q f(z) + (1-\lambda)f(z)} - 1\right)\right] =$$

$$1 + \frac{1}{b}(1 + i\tan\beta)\left\{\frac{([2]_q - 1)a_2z^2 + ([3]_q - 1)a_3z^3 + [3]_q([2]_q - 1)\lambda a_3z^3 + \lambda a_2z^2 + \lambda a_3z^3 + \ldots}{z + [2]_q\lambda a_2z^2 + [3]_q\lambda a_3z^3 + a_2z^2 + a_3z^3 - \lambda a_2z^2 - \lambda a_3z^3 + \ldots}\right\}. \tag{19}$$

Comparing the coefficients of z^2 and z^3 in (18) and (19), we obtain

$$|a_2| = \frac{|b|B_1}{4\left[\left(\left([2]_q - 1\right) + \lambda\right)\right]\left(1 + i\tan\beta\right)}, \tag{20}$$

$$\begin{aligned} |a_3| = & \frac{|b^2|B_1^2\left(1 + \left([2]_q - 1\right)\lambda\right)}{16\left[\left([2]_q - 1\right) + \lambda\right]\left[\left([3]_q - 1\right) + \left(1 + [3]_q[2]_q - [3]_q\right)\lambda\right]\left(1 + i\tan\beta\right)^2} \\ & + \frac{|b|(B_2 - B_1)}{16\left[\left([3]_q - 1\right) + \left(1 + [3]_q[2]_q - [3]_q\right)\lambda\right]\left(1 + i\tan\beta\right)}. \end{aligned} \tag{21}$$

□

As $q \to 1^-$ in the above Theorem we get the following:

Corollary 2.1. *[11] If $f \in \mathcal{M}_\lambda(b, \varphi, \Phi_{n,m})$ given by(1), then*

$$\begin{aligned} |a_2| &\leq \frac{|b|B_1}{4(1+\lambda)(1 + i\tan\beta)} \\ |a_3| &= \frac{|b^2|B_1^2}{32(1 + 2\lambda)\left(1 + i\tan\beta\right)^2} + \frac{|b|(B_2 - B_1)}{32(1 + 2\lambda)\left(1 + i\tan\beta\right)}. \end{aligned}$$

Corollary 2.2. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_1(b, \varphi, \Phi_{n,m}, q)$ then*

$$\begin{aligned} |a_2| &= \frac{|b|B_1}{4(([2]_q - 1) + 1)(1 + i\tan\beta)}, \\ |a_3| &= \frac{|b^2|B_1^2}{16\left[[3]_q[2]_q)\right]\left(1 + i\tan\beta\right)^2} + \frac{|b|(B_2 - B_1)}{16\left[([3]_q[2]_q)\right]\left(1 + i\tan\beta\right)}. \end{aligned} \tag{22}$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.3. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_1(b, \varphi, \Phi_{n,m},)$ then*

$$\begin{aligned} |a_2| &= \frac{|b|B_1}{8(1 + i\tan\beta)}, \\ |a_3| &= \frac{|b^2|B_1^2}{96\left(1 + i\tan\beta\right)^2} + \frac{|b|(B_2 - B_1)}{96(1 + i\tan\beta)}. \end{aligned} \tag{23}$$

Corollary 2.4. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_0(b, \varphi, \Phi_{n,m}, q)$ then*

$$\begin{aligned} |a_2| &= \frac{|b|B_1}{4([2]_q - 1)(1 + i\tan\beta)}, \\ |a_3| &= \frac{|b^2|B_1^2}{16([2]_q - 1)([3]_q - 1)(1 + i\tan\beta)^2} + \frac{|b|(B_2 - B_1)}{16([3]_q - 1)(1 + i\tan\beta)}. \end{aligned}$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.5. *[11]If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_0(b, \varphi, \Phi_{n,m})$ then*

$$\begin{aligned} |a_2| &= \frac{|b|B_1}{4(1+i\tan\beta)}, \\ |a_3| &= \frac{|b^2|B_1^2}{32(1+i\tan\beta)^2} + \frac{|b|(B_2 - B_1)}{32(1+i\tan\beta)}. \end{aligned} \tag{24}$$

Theorem 2.2. *If $f \in \mathcal{G}_\lambda(b, \varphi, \Phi_{n,m}, q)$ given by* (12)*, then*

$$\begin{aligned} |a_2| &= \frac{|b|B_1}{4(([2]_q - 1) + [2]_q\lambda)(1+i\tan\beta)}, \\ |a_3| &= \frac{|b^2|B_1^2}{16\left(([2]_q - 1) + [2]_q\lambda\right)\left([3]_q - 1) + [3]_q[2]_q\lambda\right)(1+i\tan\beta)^2} \\ &\quad + \frac{|b|(B_2 - B_1)}{16\left([3]_q - 1) + [3]_q[2]_q\lambda\right)(1+i\tan\beta)}. \end{aligned}$$

Proof. If $f \in \mathcal{G}_\lambda(b, \varphi, \Phi_{n,m}, q)$ then

$$1 + \frac{1}{b}\left[(1+i\tan\beta)\left(\frac{z\partial_q f(z)}{f(z)} + \lambda\frac{z^2\partial_q(z\partial_q f(z)}{f(z)} - 1\right)\right] = \varphi(\omega(z)). \tag{25}$$

Define the function $\Phi(z)$ by

$$\Phi(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + ..., \tag{26}$$

or, equivalently

$$\omega(z) = \frac{\Phi(z) - 1}{\Phi(z) + 1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - .... \tag{27}$$

From the equations (5) and (27), we have

$$\varphi\left(\frac{\Phi(z) - 1}{\Phi(z) + 1}\right) = 1 + \frac{B_1}{4}z + \frac{B_2 - B_3}{16}z^2 + \frac{3B_3 - 6B_2 - B_1}{192}z^3 + ... \tag{28}$$

By simple calculations, we get

$$1 + \frac{1}{b}\left[(1+i\tan\beta)\left(\frac{z\partial_q f(z)}{f(z)} + \lambda\frac{z^2\partial_q(z\partial_q f(z)}{f(z)} - 1\right)\right] =$$

$$1 + \frac{1}{b}(1+i\tan\beta)\left\{\frac{[2]_q a_2 z^2 + [3]_q a_3 z^3 + [2]_q\lambda a_2 z^2 + [3]_q[2]_q\lambda a_3 z^3 - a_2 z^2 - a_3 z^3 + ...}{z + a_2 z^2 + a_3 z^3 + ...}\right\}. \tag{29}$$

Comparing the coefficients of z^2 and z^3 in (28) and(29), we obtain

$$|a_2| = \frac{|b|B_1}{4(([2]_q - 1) + [2]_q\lambda)(1+i\tan\beta)}, \tag{30}$$

$$|a_3| = \frac{|b^2|B_1^2}{16\left(([2]_q-1)+[2]_q\lambda\right)\left([3]_q-1)+[3]_q[2]_q\lambda\right)(1+i\tan\beta)^2} + \frac{|b|(B_2-B_1)}{16\left([3]_q-1)+[3]_q[2]_q\lambda\right)(1+i\tan\beta)}. \tag{31}$$

□

As $q \to 1^-$ in the above Theorem we get the following:

Corollary 2.6. *[11]If $f \in \mathcal{A}$ given* (1) *be in the class $\mathcal{G}_\lambda(b,\varphi,\Phi_{n,m})$ then*

$$|a_2| = \frac{|b|B_1}{4(1+2\lambda)(1+i\tan\beta)},$$
$$|a_3| = \frac{|b^2|B_1^2}{32(1+2\lambda)(1+3\lambda)(1+i\tan\beta)^2} + \frac{|b|(B_2-B_1)}{32(1+3\lambda)(1+i\tan\beta)}.$$

Corollary 2.7. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_1(b,\varphi,\Phi_{n,m},q)$ then*

$$|a_2| = \frac{|b|B_1}{4(([2]_q-1)+[2]_q)(1+i\tan\beta)},$$
$$|a_3| = \frac{|b^2|B_1^2}{16\left(([2]_q-1)+[2]_q\right)\left([3]_q-1+[3]_q[2]_q\lambda\right)(1+i\tan\beta)} + \frac{|b|(B_2-B_1)}{16\left([3]_q-1)+[3]_q[2]_q\right)(1+i\tan\beta)}.$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.8. *[11]If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_1(b,\varphi,\Phi_{n,m})$ then*

$$|a_2| = \frac{|b|B_1}{12(1+i\tan\beta)},$$
$$|a_3| = \frac{|b^2|B_1^2}{384(1+i\tan\beta)^2} + \frac{|b|(B_2-B_1)}{128(1+i\tan\beta)}.$$

Corollary 2.9. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_0(b,\varphi,\Phi_{n,m},q)$ then*

$$|a_2| = \frac{|b|B_1}{4([2]_q-1)(1+i\tan\beta)},$$
$$|a_3| = \frac{|b^2|B_1^2}{16\left([2]_q-1\right)\left([3]_q-1\right)(1+i\tan\beta)} + \frac{|b|(B_2-B_1)}{16\left([3]_q-1\right)(1+i\tan\beta)}.$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.10. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_0(b,\varphi,\Phi_{n,m})$ then*

$$|a_2| = \frac{|b|B_1}{4(1+i\tan\beta)},$$
$$|a_3| = \frac{|b^2|B_1^2}{32(1+i\tan\beta)^2} + \frac{|b|(B_2-B_1)}{32(1+i\tan\beta)}.$$

Theorem 2.3. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b^2|}{16} \left| \frac{B_2 - B_1}{b\,(([3]_q - 1)(1 + [3]_q[2]_q - [3]_q)\lambda)\,(1 + i\tan B)} \right.$$

$$\left. + \frac{B_1^2}{(1 + i\tan B)^2} \left(\frac{(1 + ([2]_q - 1)\lambda)^2}{[((([2]_q - 1) + \lambda)]^2\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\,\lambda]} - \frac{\mu}{[([2]_q - 1) + \lambda]^2} \right) \right|.$$

Proof. From (20) and (21) we get

$$a_3 - \mu a_2^2 = \frac{b^2 B_1^2\,(1 + ([2]_q - 1)\lambda)^2}{16\,[([2]_q - 1) + \lambda]^2\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\lambda]\,(1 + i\tan\beta)^2}$$

$$+ \frac{b(B_2 - B_1)}{16\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\lambda]\,(1 + i\tan\beta)} - \frac{bB_1}{4\,[((([2]_q - 1) + \lambda)]\,(1 + i\tan\beta)}.$$

By simple calculation, we get

$$a_3 - \mu a_2^2 = \frac{b^2}{16} \left[\frac{B_2 - B_1}{b\,(([3]_q - 1)(1 + [3]_q[2]_q - [3]_q)\lambda)\,(1 + i\tan B)} \right.$$

$$\left. + \frac{B_1^2}{(1 + i\tan B)^2} \left(\frac{(1 + ([2]_q - 1)\lambda)^2}{[(([2]_q - 1) + \lambda)]^2\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\,\lambda]} - \frac{\mu}{[([2]_q - 1) + \lambda]^2} \right) \right].$$

Hence, we have

$$|a_3 - \mu a_2^2| \leq \frac{|b^2|}{16} \left| \frac{B_2 - B_1}{b\,(([3]_q - 1)(1 + [3]_q[2]_q - [3]_q)\lambda)\,(1 + i\tan B)} \right.$$

$$\left. + \frac{B_1^2}{(1 + i\tan B)^2} \left(\frac{(1 + ([2]_q - 1)\lambda)^2}{[((([2]_q - 1) + \lambda)]^2\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\,\lambda]} - \frac{\mu}{[([2]_q - 1) + \lambda]^2} \right) \right|.$$

□

As $q \to 1^-$ in the above Theorem we get the following:

Corollary 2.11. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b^2|}{32} \left| \frac{B_2 - B_1}{b\,(1 + 2\lambda)\,(1 + i\tan\beta)} + \frac{B_1^2}{(1 + i\tan B)^2} \left(\frac{1}{1 + 2\lambda} - \frac{2\mu}{[1 + \lambda]^2} \right) \right|.$$

For taking $\mu = 1$ in Theorem 2.3, we get the following:

Corollary 2.12. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{M}_\lambda(b, \varphi, \Phi_{n,m}, q)$*

$$|a_3 - a_2^2| \le \frac{|b^2|}{16}\left|\frac{B_2 - B_1}{b\,(([3]_q - 1)(1 + [3]_q[2]_q - [3]_q)\lambda)\,(1 + i\tan B)} + \frac{B_1^2}{(1 + i\tan B)^2}\left(\frac{(([2]_q - 1) + \lambda)^2 - [([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\,\lambda]}{[(1 + ([2]_q - 1)\lambda)]^2\,[([3]_q - 1) + (1 + [3]_q[2]_q - [3]_q)\,\lambda]}\right)\right|.$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.13. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m})$, then*

$$|a_3 - a_2^2| \le \frac{|b^2|}{32}\left|\frac{B_2 - B_1}{b\,(1 + 2\lambda)\,(1 + i\tan\beta)} + \frac{B_1^2}{(1 + i\tan\beta)^2}\left(\frac{1 + 2\lambda - \lambda^2}{(1 + 2\lambda)(1 + \lambda)^2}\right)\right|.$$

Theorem 2.4. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m}, q)$, then*

$$|a_3 - \mu a_2^2| \le \frac{|b^2|}{16}\left|\frac{B_2 - B_1}{b\,(([3]_q - 1) + [3]_q[2]_q\lambda)\,(1 + i\tan\beta)} + \frac{B_1^2}{(([2]_q - 1) + [2]_q\lambda)\,(1 + i\tan\beta)^2}\left(\frac{1}{(([3]_q - 1) + [3]_q[2]_q\lambda)} - \frac{\mu}{(([2]_q - 1) + [2]_q\lambda)}\right)\right|.$$

Proof. From (30) and (31), we get

$$a_3 - \mu a_2^2 = \frac{b^2 B_1^2}{16\,(([2]_q - 1) + [2]_q\lambda)\,(([3]_q - 1) + [3]_q[2]_q\lambda)\,(1 + i\tan\beta)^2} + \frac{b(B_2 - B_1)}{16\,(([3]_q - 1) + [3]_q[2]_q\lambda)\,(1 + i\tan\beta)} - \mu\frac{b^2 B_1^2}{16\,(([2]_q - 1) + [2]_q\lambda)^2\,(1 + i\tan\beta)^2}.$$

By simple calculation, we get

$$a_3 - \mu a_2^2 = \frac{b^2}{16}\left[\frac{B_2 - B_1}{b\,(([3]_q - 1) + [3]_q[2]_q\lambda)\,(1 + i\tan\beta)} + \frac{B_1^2}{(([2]_q - 1) + [2]_q\lambda)\,(1 + i\tan\beta)^2}\left(\frac{1}{(([3]_q - 1) + [3]_q[2]_q\lambda)} - \frac{\mu}{(([2]_q - 1) + [2]_q\lambda)}\right)\right].$$

Hence, we have

$$|a_3 - \mu a_2^2| \le \frac{|b^2|}{16}\left|\frac{B_2 - B_1}{b\,(([3]_q - 1) + [3]_q[2]_q\lambda)\,(1 + i\tan\beta)} + \frac{B_1^2}{(([2]_q - 1) + [2]_q\lambda)\,(1 + i\tan\beta)^2}\left(\frac{1}{(([3]_q - 1) + [3]_q[2]_q\lambda)} - \frac{\mu}{(([2]_q - 1) + [2]_q\lambda)}\right)\right|.$$

□

As $q \to 1^-$ in the above Theorem we get the following:

Corollary 2.14. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m})$, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b^2|}{32}\left|\frac{B_2 - B_1}{b\,(1+3\lambda)\,(1+i\tan\beta)} + \frac{B_1^2}{(1+2\lambda)\,(1+i\tan\beta)^2}\left(\frac{1}{(1+3\lambda)} - \frac{2\mu}{(1+2\lambda)}\right)\right|.$$

For taking $\mu = 1$, we get

Corollary 2.15. *If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m}, q)$, then*

$$|a_3 - a_2^2| \leq \frac{|b^2|}{16}\left|\frac{B_2 - B_1}{b\,(([3]_q - 1) + [3]_q([3]_q - 1)\lambda)\,(1+i\tan\beta)}\right.$$

$$\left. + \frac{B_1^2 \cos^2\beta}{(([2]_q - 1) + [2]_q([2]_q - 1)\lambda)}\left(\frac{1}{(([3]_q - 1) + [3]_q([3]_q - 1)\lambda)} - \frac{1}{(([2]_q - 1) + [2]_q([2]_q - 1)\lambda)}\right)\right|.$$

As $q \to 1^-$ in the above Corollary we get the following:

Corollary 2.16. *[11] If $f \in \mathcal{A}$ given by* (1) *be in the class $\mathcal{G}_\lambda(b, \varphi, \Phi_{n,m})$, then*

$$|a_3 - a_2^2| \leq \frac{|b^2|}{32}\left|\frac{B_2 - B_1}{b\,(1+3\lambda)\,(1+i\tan\beta)} + \frac{B_1^2}{(1+2\lambda)\,(1+i\tan\beta)^2}\left(\frac{1+4\lambda}{(1+2\lambda)(1+3\lambda)}\right)\right|.$$

REFERENCES

[1] O.A.Fadipe-Joseph, A.T.Oladipo and U.A.Ezeafulukwe, Modified sigmoid function in univalent function theory, Int. J. of Math. Sci. and Engineering Appl.,**7**(7)(2013), 313-317.

[2] M.Fekete and G.Szeg$\ddot{O}$, Eine bemerkung uber ungerade schlichte funktionen, J. Lond. Math. Soc.,**8**(1933), 85-89.

[3] F.H.Jackson, *On q-functions and a certain difference operator,* Trans. Royal Soc. Edinburgh,**46**(1909), 253-281.

[4] F.R.Keogh and E.P.Merkes, A coefficient inequality for certain classes of analytic functions. Proceedings of the American Math. Soc.,**20**(1)(1969), 8-12.

[5] G.Murugusundaramoorthy and T.Janani, "SIGMOID FUNCTION IN THE SPACE OF UNIVALENT λ-PSEUDO STARLIKE FUNCTIONS." Int. J. of Pure and Appl. Math.,**101**(1)(2015), 33-41.

[6] A.T.Oladipo, Coefficient inequality for subclass of analytic univalent functions related to simple logistic activation functions. Stud. Univ. Babes-Bolyai Math.,**61**(2016), 45-52.

[7] S.O.Olatunji, Sigmoid function in the space of univalent lambda pseudo starlike function with Sakaguchi type functions. J. of Progressive Research in Math.,**7**(4)(2016), 1164-1172.

[8] S.O.Olatunji and E.J.Dansu , Coefficient Estimates for Bazilevi Ma-Minda Functions in the Space of Sigmoid Function,Malaya J. Math.**4**(3)(2016), 505-512

[9] S.O.Olatunji, E.J.Dansu and A.Abidemi, On a Sakaguchi type class of analytic functions associated with quasi-subordination in the space of modified sigmoid functions. Electronic J. Math. Analy. Appl.,**5**(1)(2017), 97-105.

[10] C.Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht,Gottingen (1975).

[11] C.Ramachandran and K.Dhanalakshmi, Coefficients estimates for a class of spirallike function in the space of sigmoid function,Global J., of Pure and Appl. Math. Vol.,**13**(1)(2016).

[12] L.Spacek, Prispěvek k teorii funkei prostych, Čapopis Pest. Math. Fys.,**62**(1933), 12-19.