International Journal of Mathematics Research.
ISSN 0976-5840 Volume 11, Number 1 (2019), pp. 19-28

http://www.irphouse.com

Pentagonal Fuzzy Number by Cholesky Decomposition and Singular Value Decomposition

Dr. V.Vijayalakshmi[1], A.Karpagam[2]

[1,2] Department of Mathematics,

[1,2] SRM Valliammai Engineering College, Kattankulathur, India

Abstract

In this article, Cholesky decomposition Algorithm and Singular Value Decomposition Algorithm is used to solve fully fuzzy linear system of equation with pentagonal fuzzy number as inputs has been introduced. An algorithms for n x n fully fuzzy linear system $\bar{A}\otimes\bar{x} = \bar{b}$ where $\bar{A}$ is a fuzzy matrix, $\bar{x} and \bar{b}$ are fuzzy vectors.Algorithms have been introduced and the numerical examples have been solved.

Keywords: Fully Fuzzy Linear System, Pentagonal fuzzy number, Cholesky decomposition method,Singular Value Decomposition Method

Mathematical Review Subject Classification: Fuzzy algebra 08A72, Algebraic system of Matrices 15A30, Linear Equations 15A06.

1. INTRODUCTION

L. A. Zadeh[6] introduced the concepts of fuzzy numbers and fuzzy arithmetic. Fuzzy metric space are used to solve fuzzy metric spaces, fuzzy differential equations, fuzzy linear and non linear system etc., A. Kumar et,al [1,2,3] introduced fully fuzzy linear system with arbitrary coefficients, A New Approach for Solving Fully Fuzzy Linear Systems, A New Computational Method for Solving Fully Fuzzy Linear

Systems of Triangular Fuzzy Numbers,

Linear system of equations has widely applied in many areas of science, Economics, Management and Engineering. Fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number was first proposed by Friedman.et.al.

Dehgan.et.al [7,8,9] introduced fully fuzzy linear system of the form Ax + b, Computational method for solving fully fuzzy linear systems, solving using iterative techniques.

Vijayalakshmi V.et.al [13,14] introduced the concepts of solving FFLS for triangular, trapezoidal, hexagonal and octagonal fuzzy numbers.G. Malkawi, N. Ahmad and H. Ibrahim,[5]investigated, "Solving fully fuzzy linear systems by using implicit gauss cholesky algorithm",

A new computational method to solve FFLS by relying on the computation of row reduced echelon form. GhassanMalkawi, Nazihah Ahmad anHaslinda Ibrahim [4], Solving fully fuzzy linear system with the necessary and sufficient condition to have a positive solution, Applied Mathematics & Information Sciences.ThangarajBeaula and L.Mohan[12] introduced "Cholesky Decomposition Method for solving FFLS for trapezoidal fuzzy number".

S. Abbasbandy and M. S. Has hemi[11] proposed solving Fully Fuzzy Linear Systems by Using Implicit Gauss–Cholesky Algorithm

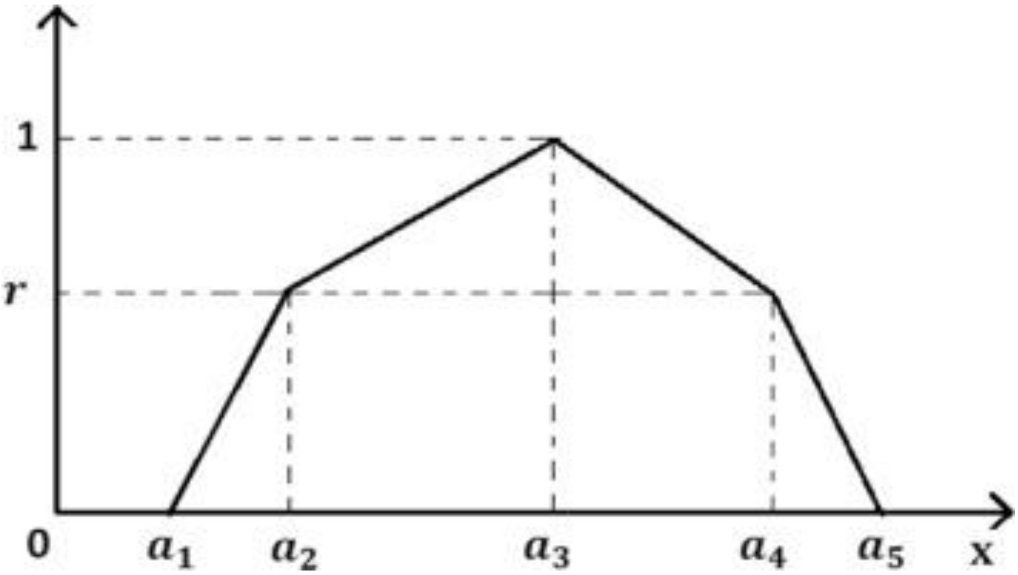
This article mainly consists of preliminary definitions with their diagrammatic representation in section2. Section 3 proposed a new algorithm solve fuzzy linear system in cholesky decompositionand Singular Value Decomposition Algorithm. In section4 numerical examples have been illustrated by solving using cholesky decomposition method and Singular Value Decomposition method. In section 5, conclusion about the results is established.

2. PRELIMINARIES

2.1 DEFINITION: FUZZY SET

Let X be a non empty set. A fuzzy set A is characterized by its membership function A : X→ [0,1] and A(x) is interpreted as the degree of membership of element in fuzzy A for each $x \epsilon X$. The value 0 represent non membership and the value 1 represent membership in between values represent intermediate degrees of membership.

2.2 Definition: Pentagonal Fuzzy Number



A Pentagon Fuzzy Number $\tilde{A}_p = (a_1, a_2, a_3, a_4, a_5)$ Where a_1, a_2, a_3, a_4 and a_5 are real numbers and $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ with membership function is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & for \quad x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & for \; a_1 \leq x \leq a_2 \\ \dfrac{x - a_2}{a_3 - a_2} & for \; a_2 \leq x \leq a_3 \\ 1 & for \quad x = a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & for \; a_3 \leq x \leq a_4 \\ \dfrac{a_5 - x}{a_5 - a_4} & for \; a_4 \leq x \leq a_5 \\ 0 & for \quad x > a_5 \end{cases}$$

3. PROPOSED METHOD FOR PENTAGONAL FUZZY NUMBER

3.1 CHOLESKY DECOMPOSITION ALGORITHM

Any fuzzy linear system of equations in the form of pentagonal fuzzy matrices that can be decomposed into the form such that A = $L_1 L_1^T$ where A, B, C, D, E is the symmetric and positive definite and L is a lower triangular matrix. To find L_1, L_2, L_3, L_4, L_5 by Cholesky decomposition algorithm is used.

Consider fully fuzzy linear systems,

Step I

$\bar{A} \otimes \bar{x} = \bar{b}$where$\bar{A}$ = (A,B,C,D,E) ,

$\bar{b}$ = (b,h,g,l,m)≥ 0 , $\bar{x}$= (x,y,z,u,v)≥ 0

(A, B,C,D,E) ⊗ (x,y,z,u,v) = (b , h , g, l,m)

Step II

$(L_1L_1^T, L_2L_2^T, L_3L_3^T, L_4L_4^T, L_5L_5^T) \otimes$ (x,y,z,u,v) = (b, h , g, l ,m)

$(L_1L_1^T\ x, L_2L_2^Ty, L_3L_3^T z, L_4L_4^Tu, L_5L_5^T v)$ = (b , h , g , l, m)

Step III

$L_1L_1^T\ x = b$, $L_2L_2^Ty = h$, $L_3L_3^T z = g$, $L_4L_4^Tu = l$, $L_5L_5^T v = m$

Therefore $x = L_1^{T-1}L_1^{-1}b$

$y = L_2^{T-1}L_2^{-1}h$, $z = L_3^{T-1}L_3^{-1}g$,

u= $L_4^{T-1}L_4^{-1}l$

$v = L_5^{T-1}L_5^{-1}m$

3.2SINGULAR VALUE DECOMPOSITIONALGORITHM

Any fuzzy linear system of equations in the form of pentagonal fuzzy matrices that can be decomposed into the form such that A = $USV^T$where A, B, C, D, E is the rectangular matrix can be decompose into an orthogonal matrix U , Diagonal matrix S and the transpose of an orthogonal matrix V.

Consider fully fuzzy linear systems,

Step I

$\bar{A} \otimes \bar{x} = \bar{b}$where$\bar{A}$ = (A,B,C,D,E) ,

$\bar{b}$ = (b,h,g,l,m)≥ 0 , $\bar{x}$= (x,y,z,u,v)≥ 0

(A, B,C,D,E) ⊗ (x,y,z,u,v) = (b , h , g, l,m)

Step II

$(U_1S_1V_1^T, U_2S_2V_2^T, U_3S_3V_3^T, U_4S_4V_4^T, U_5S_5V_5^T) \otimes$ (x,y,z,u,v) = (b, h , g, l ,m)

$(U_1S_1V_1^Tx, U_2S_2V_2^Ty,, U_3S_3V_3^T$ z $, U_4S_4V_4^Tu, U_5S_5V_5^Tv)$= (b , h , g , l, m)

Step III

$U_1S_1V_1^Tx=$ b,$U_2S_2V_2^Ty = h$, ,$U_3S_3V_3^T$ z = g, $U_4S_4V_4^Tu = l$,$U_5S_5V_5^Tv = m$

Therefore $x = V_1^{T-1}S_1^{-1}U_1^{-1}$b

$$y = V_2^{T-1}S_2^{-1}U_2^{-1}\text{h}$$
$$z = V_3^{T-1}S_3^{-1}U_3^{-1}\text{g}$$
$$u = V_4^{T-1}S_4^{-1}U_4^{-1}\text{l}$$
$$v = V_5^{T-1}S_5^{-1}U_5^{-1}\text{m}$$

4.NUMERICAL EXAMPLE

Ex 4.1 Solve the following FFLS by Cholesky Decomposition Method.

$$\begin{bmatrix} (3,6,8,9,10) & (1,6,7,9,12) & (1,3,4,9,12) \\ (1,6,7,9,12) & (2,8,10,12,15) & (6,8,9,11,15) \\ (1,3,4,9,12) & (6,8,9,11,15) & (20,21,22,23,24) \end{bmatrix} \begin{bmatrix} (x_1,y_1,z_1,u_1,v_1) \\ (x_2,y_2,z_2,u_2,v_2) \\ (x_3,y_3,z_3,u_3,v_3) \end{bmatrix}$$

$$= \begin{bmatrix} (12,20,23,25,28) \\ (10,14,17,21,23) \\ (18,21,23,26,29) \end{bmatrix}$$

Solution :

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 6 \\ 1 & 6 & 20 \end{bmatrix} B = \begin{bmatrix} 6 & 6 & 3 \\ 6 & 8 & 8 \\ 3 & 8 & 21 \end{bmatrix} C = \begin{bmatrix} 8 & 7 & 9 \\ 7 & 10 & 9 \\ 4 & 9 & 22 \end{bmatrix}$$

$$D = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 12 & 11 \\ 9 & 11 & 23 \end{bmatrix} \quad E = \begin{bmatrix} 10 & 12 & 12 \\ 12 & 15 & 15 \\ 12 & 15 & 24 \end{bmatrix}$$

$A = L_1L_1^T$

$$\text{Where} L_1 = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{5}}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{17}{\sqrt{15}} & 0.632 \end{bmatrix} L_1^T = \begin{bmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{\sqrt{3}} & \frac{17}{\sqrt{15}} \\ 0 & 0 & 0.632 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 0.5714 & 0 & 0 \\ -0.2582 & 0.7746 & 0 \\ 1.2658 & -5.3797 & 1.5823 \end{bmatrix} L_1^{T-1} = \begin{bmatrix} 0.5714 & -0.2582 & 1.2658 \\ 0 & 0.7746 & -5.3797 \\ 0 & 0 & 1.5823 \end{bmatrix}$$

$$x = L_1^{T-1} L_1^{-1} b = \begin{bmatrix} -10.0185 \\ 58.0784 \\ -16.0231 \end{bmatrix}$$

Similarlyy = $\begin{bmatrix} 6.9771 \\ -4.1646 \\ 1.0417 \end{bmatrix}$ z = $\begin{bmatrix} 3.6020 \\ -1.2426 \\ 0.7207 \end{bmatrix}$ u = $\begin{bmatrix} 4.0146 \\ -1.5263 \\ 0.2895 \end{bmatrix}$ v = $\begin{bmatrix} 24.3689 \\ -19.0304 \\ 1.0563 \end{bmatrix}$

Ex 4.2 Solve the following FFLS by singular value decompositionMethod.

$$\begin{bmatrix} (3,6,8,9,10) & (1,6,7,9,12) & (1,3,4,9,12) \\ (1,6,7,9,12) & (2,8,10,12,15) & (6,8,9,11,15) \\ (1,3,4,9,12) & (6,8,9,11,15) & (20,21,22,23,24) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, u_1, v_1) \\ (x_2, y_2, z_2, u_2, v_2) \\ (x_3, y_3, z_3, u_3, v_3) \end{bmatrix}$$

$$= \begin{bmatrix} (12,20,23,25,28) \\ (10,14,17,21,23) \\ (18,21,23,26,29) \end{bmatrix}$$

Solution :

A = $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 6 \\ 1 & 6 & 20 \end{bmatrix}$ B = $\begin{bmatrix} 6 & 6 & 3 \\ 6 & 8 & 8 \\ 3 & 8 & 21 \end{bmatrix}$ C = $\begin{bmatrix} 8 & 7 & 9 \\ 7 & 10 & 9 \\ 4 & 9 & 22 \end{bmatrix}$

D = $\begin{bmatrix} 9 & 9 & 9 \\ 9 & 12 & 11 \\ 9 & 11 & 23 \end{bmatrix}$ E = $\begin{bmatrix} 10 & 12 & 12 \\ 12 & 15 & 15 \\ 12 & 15 & 24 \end{bmatrix}$

$A = U_1 S_1 V_1^T x$

Where

$$U_1 = \begin{bmatrix} -0.066 & 0.972 & 0.224 \\ -0.291 & 0.196 & -0.936 \\ -0.954 & -0.127 & 0.270 \end{bmatrix} S_1 = \begin{bmatrix} 21.899 & 0 & 0 \\ 0 & 3.071 & 0 \\ 0 & 0 & 0.030 \end{bmatrix}$$

$$V_1^T = \begin{bmatrix} -0.066 & -0.291 & -0.954 \\ 0.972 & 0.196 & -0.127 \\ 0.224 & -0.936 & 0.270 \end{bmatrix}$$

$$V_1^{T-1} = \begin{bmatrix} -0.066 & 0.972 & 0.224 \\ -0.291 & 0.196 & -0.936 \\ -0.954 & -0.127 & 0.270 \end{bmatrix}$$

$$S_1^{-1} = \begin{bmatrix} 0.046 & 0 & 0 \\ 0 & 0.326 & 0 \\ 0 & 0 & 33.333 \end{bmatrix}$$

$$U_1^{-1} = \begin{bmatrix} -0.066 & -0.291 & -0.954 \\ 0.972 & 0.196 & -0.127 \\ 0.224 & -0.936 & 0.270 \end{bmatrix}$$

$$x = V_1^{T-1} S_1^{-1} U_1^{-1} \text{b} = \begin{bmatrix} -10.0185 \\ 58.0784 \\ -16.0231 \end{bmatrix}$$

Similarly $y = \begin{bmatrix} 6.9771 \\ -4.1646 \\ 1.0417 \end{bmatrix}$ $z = \begin{bmatrix} 3.6020 \\ -1.2426 \\ 0.7207 \end{bmatrix}$ $u = \begin{bmatrix} 4.0146 \\ -1.5263 \\ 0.2895 \end{bmatrix}$ $v = \begin{bmatrix} 24.3689 \\ -19.0304 \\ 1.0563 \end{bmatrix}$

4.3 Comparison Results in Cholesky and SVD

Sl No	Problems	Cholesky	SVD
1	$\begin{bmatrix} (3,6,8,9,10) & (1,6,7,9,12) & (1,3,4,9,12) \\ (1,6,7,9,12) & (2,8,10,12,15) & (6,8,9,11,15) \\ (1,3,4,9,12) & (6,8,9,11,15) & (20,21,22,23,24) \end{bmatrix}$ * $\begin{bmatrix} (x_1,y_1,z_1,u_1,v_1) \\ (x_2,y_2,z_2,u_2,v_2) \\ (x_3,y_3,z_3,u_3,v_3) \end{bmatrix} = \begin{bmatrix} (12,20,23,25,28) \\ (10,14,17,21,23) \\ (18,21,23,26,29) \end{bmatrix}$	$x = \begin{bmatrix} -10.0185 \\ 58.0784 \\ -16.0231 \end{bmatrix}$ $y = \begin{bmatrix} 6.9771 \\ -4.1646 \\ 1.0417 \end{bmatrix}$ $z = \begin{bmatrix} 3.6020 \\ -1.2426 \\ 0.7207 \end{bmatrix}$ $u = \begin{bmatrix} 4.0146 \\ -1.5263 \\ 0.2895 \end{bmatrix}$ $v = \begin{bmatrix} 24.3689 \\ -19.0304 \\ 1.0563 \end{bmatrix}$	$x = \begin{bmatrix} -10.0185 \\ 58.0784 \\ -16.0231 \end{bmatrix}$ $y = \begin{bmatrix} 6.9771 \\ -4.1646 \\ 1.0417 \end{bmatrix}$ $z = \begin{bmatrix} 3.6020 \\ -1.2426 \\ 0.7207 \end{bmatrix}$ $u = \begin{bmatrix} 4.0146 \\ -1.5263 \\ 0.2895 \end{bmatrix}$ $v = \begin{bmatrix} 24.3689 \\ -19.0304 \\ 1.0563 \end{bmatrix}$

Sl No	Problems	Cholesky	SVD
2	$\begin{bmatrix} (1,2,8,9,10) & (1,3,8,10,11) & (4,5,7,8,12) \\ (1,3,8,10,11) & (4,6,9,12,13) & (3,6,8,11,13) \\ (4,5,7,8,12) & (3,6,8,11,13) & (17,19,20,22,32) \end{bmatrix} *$ $\begin{bmatrix} (x_1,y_1,z_1,u_1,v_1) \\ (x_2,y_2,z_2,u_2,v_2) \\ (x_3,y_3,z_3,u_3,v_3) \end{bmatrix} = \begin{bmatrix} (10,15,18,21,24) \\ (12,14,19,22,26) \\ (16,17,21,23,28) \end{bmatrix}$	x = $\begin{bmatrix} 161.22 \\ -11.002 \\ -35.034 \end{bmatrix}$ y = $\begin{bmatrix} -26.115 \\ 60.739 \\ 77.981 \end{bmatrix}$ z = $\begin{bmatrix} 1.304 \\ 0.664 \\ 0.336 \end{bmatrix}$ u = $\begin{bmatrix} 4.821 \\ -2.622 \\ 0.459 \end{bmatrix}$ v = $\begin{bmatrix} 6.116 \\ -0.3461 \\ -0.151 \end{bmatrix}$	x = $\begin{bmatrix} 161.22 \\ -11.002 \\ -35.034 \end{bmatrix}$ y = $\begin{bmatrix} -26.115 \\ 60.739 \\ 77.981 \end{bmatrix}$ z = $\begin{bmatrix} 1.304 \\ 0.664 \\ 0.336 \end{bmatrix}$ u = $\begin{bmatrix} 4.821 \\ -2.622 \\ 0.459 \end{bmatrix}$ v = $\begin{bmatrix} 6.116 \\ -0.3461 \\ -0.151 \end{bmatrix}$
3	$\begin{bmatrix} (1,3,7,10,11) & (1,3,5,8,9) & (3,6,8,12,13) \\ (1,3,5,8,9) & (9,6,8,11,14) & (5,7,8,10,15) \\ (3,6,8,12,13) & (5,7,8,10,15) & (21,23,24,26,31) \end{bmatrix} *$ $\begin{bmatrix} (x_1,y_1,z_1,u_1,v_1) \\ (x_2,y_2,z_2,u_2,v_2) \\ (x_3,y_3,z_3,u_3,v_3) \end{bmatrix} = \begin{bmatrix} (10,12,14,18,21) \\ (14,16,17,19,20) \\ (12,13,15,21,23) \end{bmatrix}$	x = $\begin{bmatrix} 13.196 \\ 2.636 \\ -1.920 \end{bmatrix}$ y = $\begin{bmatrix} 4.588 \\ 1.713 \\ -1.146 \end{bmatrix}$ z = $\begin{bmatrix} 1.130 \\ 1.747 \\ -0.335 \end{bmatrix}$ u = $\begin{bmatrix} 1.089 \\ 1.004 \\ -0.110 \end{bmatrix}$ v = $\begin{bmatrix} 1.754 \\ 0.594 \\ -0.296 \end{bmatrix}$	x = $\begin{bmatrix} 13.196 \\ 2.636 \\ -1.920 \end{bmatrix}$ y = $\begin{bmatrix} 4.588 \\ 1.713 \\ -1.146 \end{bmatrix}$ z = $\begin{bmatrix} 1.130 \\ 1.747 \\ -0.335 \end{bmatrix}$ u = $\begin{bmatrix} 1.089 \\ 1.004 \\ -0.110 \end{bmatrix}$ v = $\begin{bmatrix} 1.754 \\ 0.594 \\ -0.296 \end{bmatrix}$

5. CONCLUSION

In this article, a new methodology is introduced to obtain a same solution of FFLS in pentagonal fuzzy number by Cholesky Decomposition Algorithm and SingularValue Decomposition Algorithm. CDA is less time consumption compared to SVDA because CDA requires less number of variables. Therefore it is easier to solve fully fuzzy linear system of equations.

REFERENCES

[1] A. Kumar, A. Bansal and Neetu, “Solution of fully fuzzy linear system with arbitrary coefficients”, International Journal of Applied Mathematics and Computation, **3**, 232-237(2011).

[2] A. Kumar, Neetu and A. Bansal, “A New Approach for Solving Fully Fuzzy Linear Systems”, Hindawi Publishing Corporation, 1-8 (2011)

[3] A. Kumar, Neetu and A. Bansal, “A New Computational Method for Solving Fully Fuzzy Linear Systems of Triangular Fuzzy Numbers, Fuzzy Inf. Eng., 63-73 (2012).

[4] GhassanMalkawi, Nazihah Ahmad an Haslinda Ibrahim, Solving fully fuzzy linear system with the necessary and sufficient condition to have a positive solution, Applied Mathematics & Information Sciences, 8, No.3, 1003-1019 (2014)

[5] G. Malkawi, N. Ahmad and H. Ibrahim, A note on “Solving fully fuzzy linear systems by using implicit gauss cholesky algorithm”, Comput.Math. Model, (2013)

[6] L. A. Zadeh, Fuzzy sets., Information and Control, **8**, 338-353 (1965).

[7] M. Dehghan and B. Hashemi, Solution of the fully fuzzy linear systems using the decomposition procedure, Allied Mathematics and Computation, **182**, 1568–1580 (2006).

[8] M. Dehghan, B. Hashemi and M. Ghatee, Computational methods for solving fully fuzzy linear systems, Alied Mathematics and Computation, **179**, 328–343 (2006).

[9] M. Dehghan, B. Hashemi and M. Ghatee, Solution of the fully fuzzy linear systems using iterative techniques, Chaos, Solitons and Fractals, **34**, 316–336 (2007).

[10] M. Otadi and M. Mosleh, Solving fully fuzzy matrix equations, Allied Mathematical Modelling, **36**, 6114–6121(2012).

[11] S. Abbasbandy and M. S. Hashemi, “Solving Fully Fuzzy Linear Systems by Using Implicit Gauss–Cholesky Algorithm”, Computational Mathematics and Modeling, **1**, 535-541 (2012).

[12] ThangarajBeaula and L.Mohan, "Cholesky Decomposition Method for solving FFLS for trapezoidal fuzzy number", Fuzzy Mathematical Archieve, Vol 14, No.2,261-265 (2017)

[13] V.Vijayalakshmi, R.Sattanathan, "ST decomposition method for solving FFLS" in International Journal of Ultra Scientists of Physical sciences volume 22 Number 3(M) Aug 2011, ISSN: 0970-9150 pp 747-750 impact factor .045

[14] V. Vijayalakshmi, Dr. R. Sattanathan, Published a paper titled "ST decomposition method for solving FFLS using Gauss Jordon method for Trapezoidal fuzzy numbers" in International Mathematical Forum , Bulgaria, Europe Vol 6, 2011 No.45 ISSN: 1312-7594 pp 2245-2254.

ABOUT THE AUTHORS

Dr. V. Vijayalakshmi is currently working as a Assistant Professor in the Department of Mathematics, SRM Valliammai Engineering College(SRM group), kattankulathur, Chennai, India. She has completed M.Sc., M.Phil., PGDCA., PhD in various universities. She has 20 years of teaching and 10 years of research experience. She has cleared SET (State eligibility test) examination conducted by Goverenment of Tamil Nadu. She has more than 10 publications in reputed journals on her credit. Her area of research fuzzy logic, fuzzy mathematics, fuzzy Matrices.

Ms. A. Karpagam is currently working as a Assistant Professor in the Department of Mathematics, SRM Valliammai Engineering College(SRM group), kattankulathur, Chennai, India. She has completed M.Sc., M.Phil.,in Anna University M.I.T. She has 6 years of teaching years of research experience. She has cleared SET (State eligibility test) examination conducted by Goverenment of Tamil Nadu. She has more than 10 publications in reputed journals on her credit. Her area of research fuzzy logic, fuzzy mathematics, fuzzy Operation Research.