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A New Theorem on Orthogonal Quadrilaterals

Brian Ekanyu

Abstract

In this paper, I prove an existence of a new theorem on orthogonal quadrilateralss. The theorem is stated as:

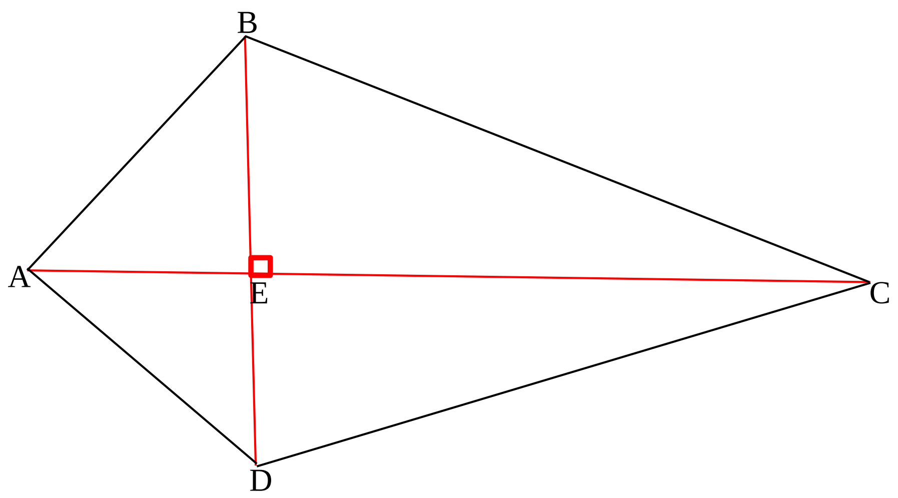
The sum of the product of side, **a** and its altimedian distance, $\boldsymbol{\alpha}$ and the product of side, **d** and its altimedian distance, $\boldsymbol{\delta}$ is equal to the sum of the product of side, **b** and its altimedian distance, $\boldsymbol{\beta}$ and the product of side, **c** and its altimedian distance, $\boldsymbol{\gamma}$ i.e.:

$$\boldsymbol{\alpha a + \delta d = \beta b + \gamma c}$$

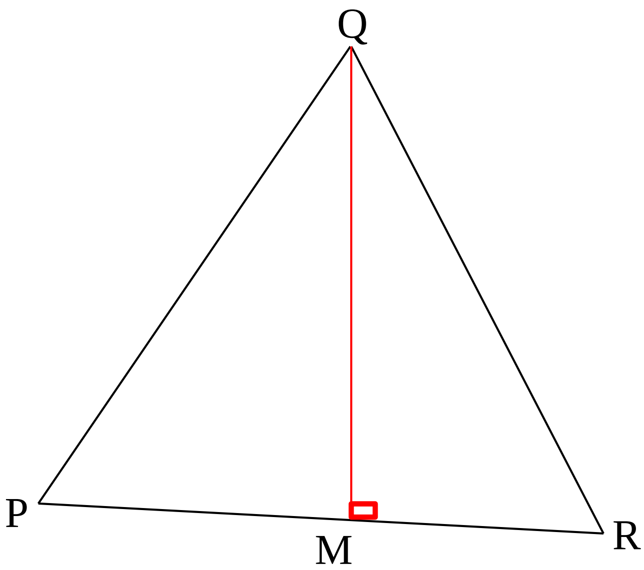
INTRODUCTION

Definitions

Orthogonal quadrilateral: Is a quadrilateral in which the diagonals cross at right angles. In other words, it is a four-sided figure in which the line segments between non-adjacent vertices are orthogonal (perpendicular) to each other.



Altitude of a triangle: Is a line segment through a vertex and perpendicular to a line containing the base (the side opposite the vertex). This line containing the opposite side is called the extended base of the altitude. The intersection of the extended base and the altitude is called the foot of the altitude. The length of the altitude, often simply called "the altitude," is the distance between the vertex and the foot of the altitude.

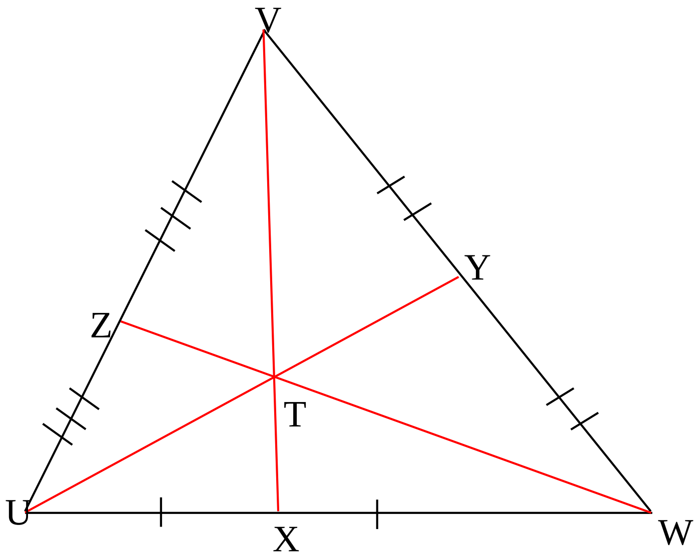


Q- Is the vertex

M- Is the foot of the altitude

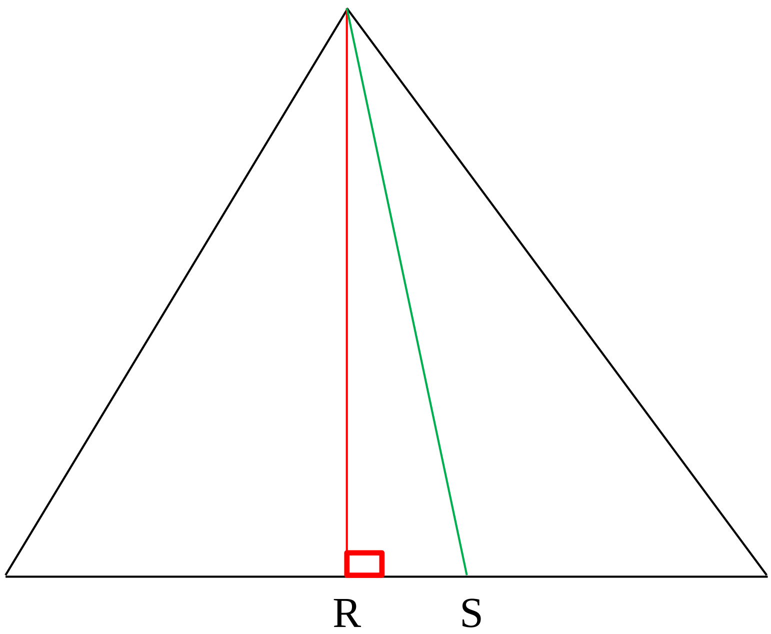
$\overline{QM}$ – Is the altitude of the triangle **PQR**

Median of a triangle: Is a line segment joining a vertex to the mid-point of the opposite side, thus bisecting that side. Every triangle has exactly three medians, one from each vertex and they all intersect each other at the triangle's centroid.

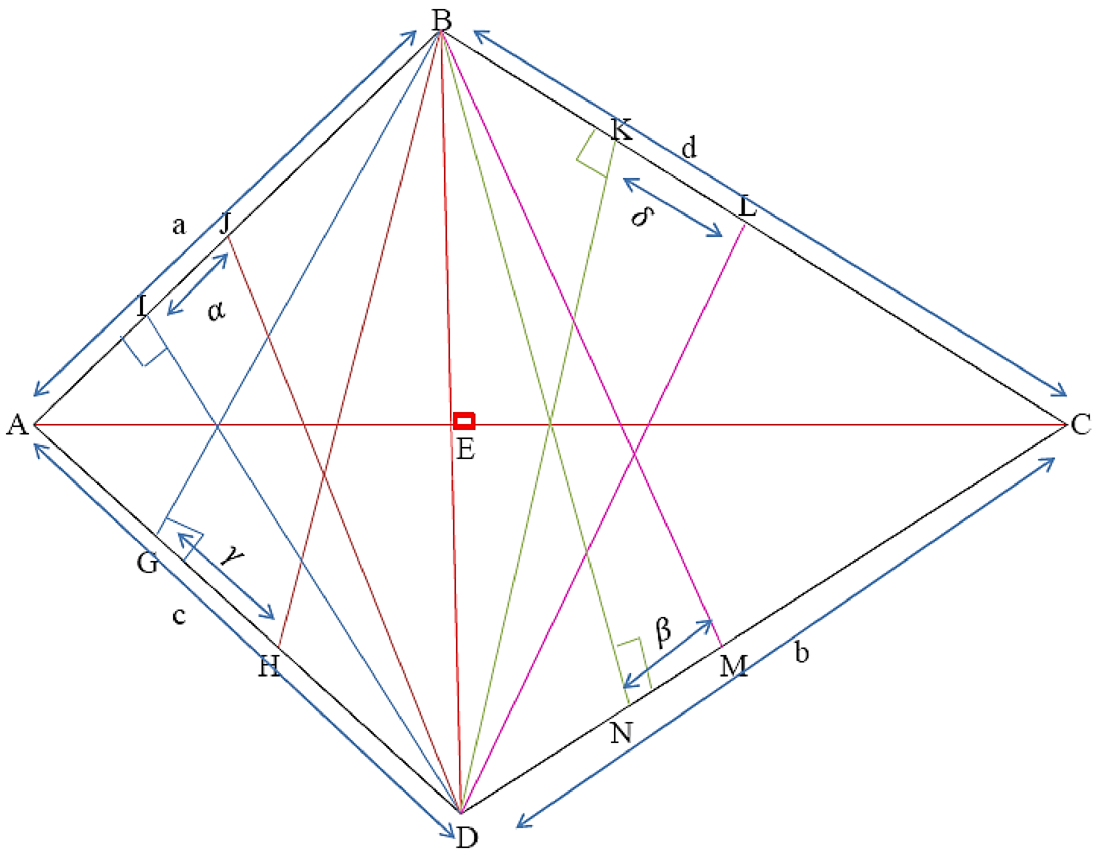


T-Is the centroid

Altimedian distance: Is the distance between the foot of the altitude and the mid-point of the base line.



$\overline{RS}$ – Is the altimedian distance



Theorem: Consider an orthogonal quadrilateral **ABCD** as shown above such that:

$\overline{AB}$=a, $\overline{IJ}$ $=\alpha$

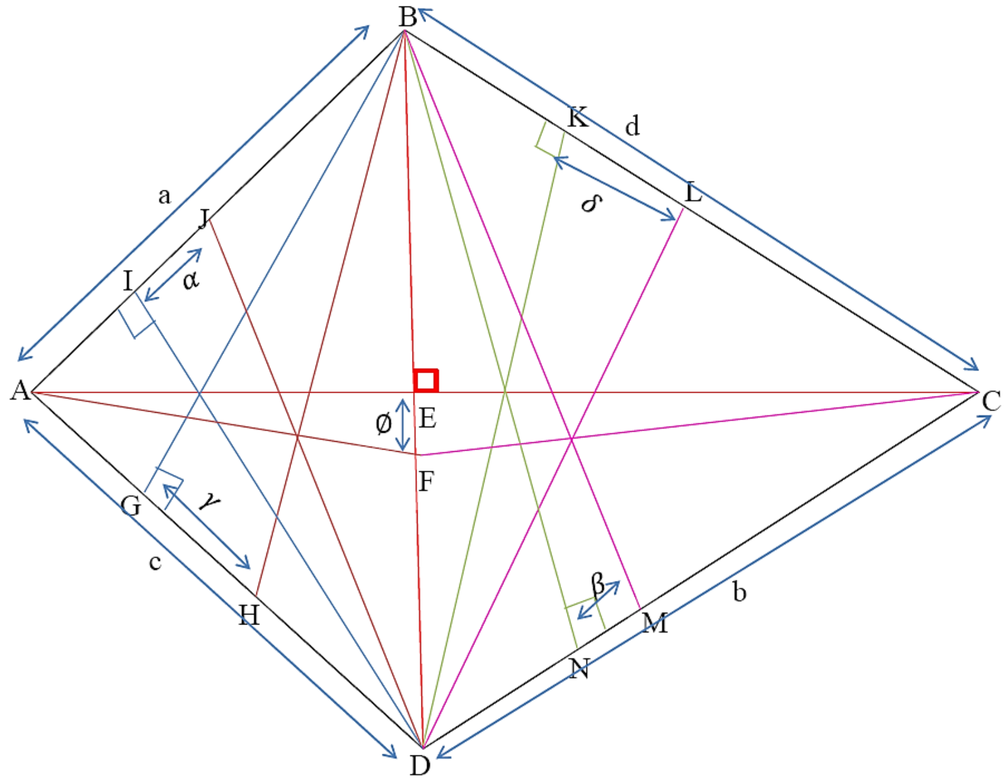
$\overline{BC} = d, \overline{KL} = \delta$

$\overline{AD} = c, \overline{GH} = \gamma$

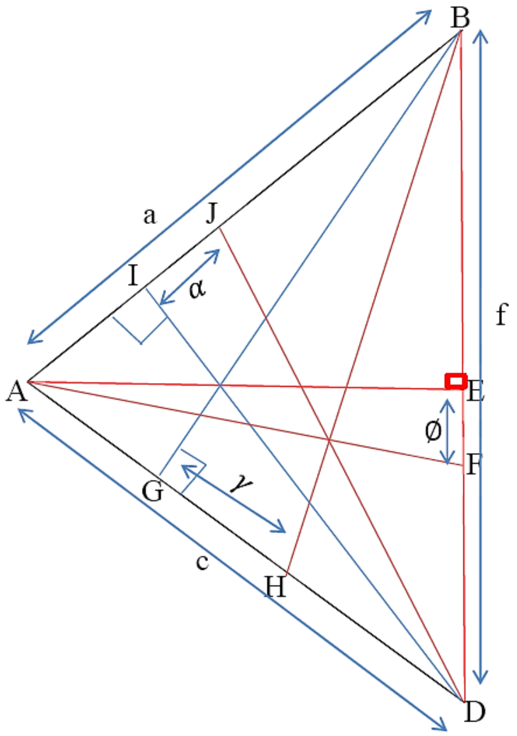
$\overline{CD} = b, \overline{MN} = \beta$

Then $\alpha a + \delta d = \beta b + \gamma c$ where α, β, γ *and* δ are altimedian distances.

Proof



From triangle **ABD**



$\overline{AB}$=a, $\overline{IJ} = \alpha$

$\overline{AD}$=c, $\overline{GH} = \gamma$

$\overline{BD}$=f, $\overline{EF} = \emptyset$

Considering triangle **ABG**

$$\overline{AG} = \left(\frac{c}{2} - \gamma\right), \overline{BG} = h_1, \overline{AB} = a$$

$$\overline{AG}^2 + \overline{BG}^2 = \overline{AB}^2$$

$$\left(\frac{c}{2} - \gamma\right)^2 + {h_1}^2 = a^2$$

$$ {h_1}^2 = a^2 - \frac{c^2}{4} + \gamma c - \gamma^2 \qquad [1]$$

Considering triangle BGD

$$\overline{BG} = h_1, \overline{GD} = \left(\frac{c}{2} + \gamma\right), \overline{BD} = f$$

$$\overline{BG}^2 + \overline{GD}^2 = \overline{BD}^2$$

$$ {h_1}^2 + \left(\frac{c}{2} + \gamma\right)^2 = f^2$$

$$ {h_1}^2 = f^2 - \frac{c^2}{4} - \gamma c - \gamma^2 \qquad [2]$$

Equating [**1**] and [**2**]

$$a^2 - \frac{c^2}{4} + \gamma c - \gamma^2 = f^2 - \frac{c^2}{4} - \gamma c - \gamma^2$$

$$2\gamma c = f^2 - a^2 \qquad [3]$$

Considering triangle **ADI**

$$\overline{AI} = \left(\frac{a}{2} - \alpha\right), \overline{DI} = h_2, \overline{AD} = c$$

$$\overline{AI}^2 + \overline{DI}^2 = \overline{AD}^2$$

$$\left(\frac{a}{2} - \alpha\right)^2 + {h_2}^2 = c^2$$

$$ {h_2}^2 = c^2 - \frac{a^2}{4} + \alpha a - \alpha^2 \qquad [4]$$

Considering triangle **BDI**

$$\overline{BI}^2 + \overline{DI}^2 = \overline{BD}^2$$

$$\overline{BI} = \left(\frac{a}{2} + \alpha\right), \overline{DI} = h_2, \overline{BD} = f$$

$$\left(\frac{a}{2} + \alpha\right)^2 + {h_2}^2 = f^2$$

$\boldsymbol{h_2}^2 = \boldsymbol{f}^2 - \frac{\boldsymbol{a}^2}{4} - \boldsymbol{\alpha a} - \boldsymbol{\alpha}^2$ [**5**]

Equating [**4**] and [**5**]

$\boldsymbol{c}^2 - \frac{\boldsymbol{a}^2}{4} + \boldsymbol{\alpha a} - \boldsymbol{\alpha}^2 = \boldsymbol{f}^2 - \frac{\boldsymbol{a}^2}{4} - \boldsymbol{\alpha a} - \boldsymbol{\alpha}^2$

$2\boldsymbol{\alpha a} = \boldsymbol{f}^2 - \boldsymbol{c}^2$ [**6**]

Considering triangle **ADE**

$\overline{AE} = h_3, \overline{DE} = \left(\frac{f}{2} + \emptyset\right), \overline{AD} = c$

$\overline{AE}^2 + \overline{DE}^2 = \overline{AD}^2$

${h_3}^2 + \left(\frac{f}{2} + \emptyset\right)^2 = c^2$

${h_3}^2 = c^2 - \frac{f^2}{4} - \emptyset f - \emptyset^2$ [**7**]

Considering triangle **ABE**

$\overline{AE} = h_3, \overline{BE} = \left(\frac{f}{2} - \emptyset\right), \overline{AB} = a$

$\overline{AE}^2 + \overline{BE}^2 = \overline{AB}^2$

${h_3}^2 + \left(\frac{f}{2} - \emptyset\right)^2 = a^2$

${h_3}^2 = a^2 - \frac{f^2}{4} + \emptyset f - \emptyset^2$ [**8**]

Equating [**7**] and [**8**]

$a^2 - \frac{f^2}{4} + \emptyset f - \emptyset^2 = c^2 - \frac{f^2}{4} - \emptyset f - \emptyset^2$

$2\emptyset f = c^2 - a^2$ [**9**]

Adding [**3**], [**6**] and [**9**]

$2\gamma c + 2\alpha a + 2\emptyset f = f^2 - a^2 + f^2 - c^2 + c^2 - a^2$

$2\alpha a + 2\gamma c + 2\emptyset f = 2(f^2 - a^2)$

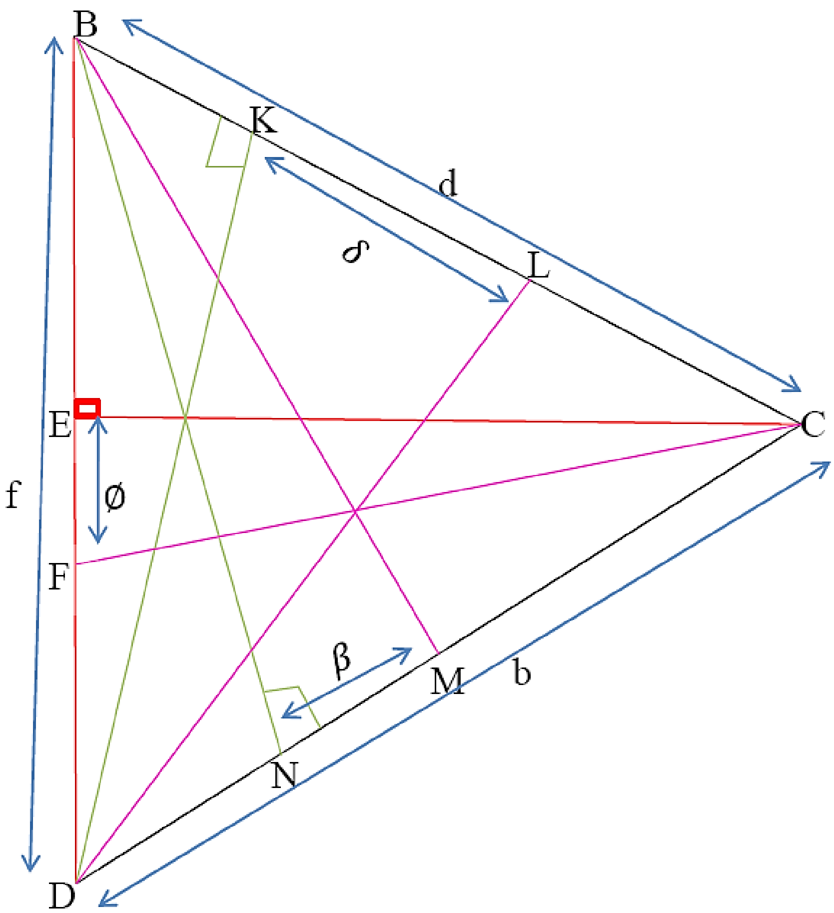
$\alpha a + \gamma c + \emptyset f = (f^2 - a^2)$

But $f^2 - a^2 = 2\gamma c$

$\alpha a + \gamma c + \emptyset f = 2\gamma c$

$\therefore\ \alpha a + \emptyset f = \gamma c$ [**10**]

From triangle **BCD**



$\overline{DC} = b, \overline{MN} = \beta$

$\overline{BC} = d, \overline{KL} = \delta$

$\overline{BD} = f, \overline{EF} = \emptyset$

Considering triangle **BDN**

$\overline{BN} = h_4, \overline{DN} = \left(\frac{b}{2} - \beta\right), \overline{BD} = f$

$\overline{BN}^2 + \overline{DN}^2 = \overline{BD}^2$

$h_4{}^2 + \left(\frac{b}{2} - \beta\right)^2 = f^2$

$$h_4{}^2 = f^2 - \frac{b^2}{4} + \beta b - \beta^2 \quad [11]$$

Considering triangle **BCN**

$\overline{BN} = h_4, \overline{CN} = \left(\frac{b}{2} + \beta\right), \overline{BC} = d$

$\overline{BN}^2 + \overline{CN}^2 = \overline{BC}^2$

$h_4{}^2 + \left(\frac{b}{2} + \beta\right)^2 = d^2$

$h_4{}^2 = d^2 - \frac{b^2}{4} - \beta b - \beta^2$ [12]

Equating [11] and [12]

$f^2 - \frac{b^2}{4} + \beta b - \beta^2 = d^2 - \frac{b^2}{4} - \beta b - \beta^2$

$2\beta b = d^2 - f^2$ [13]

Considering triangle **BDK**

$\overline{BK} = \left(\frac{d}{2} - \delta\right), \overline{DK} = h_5, \overline{BD} = f$

$\overline{BK}^2 + \overline{DK}^2 = \overline{BD}^2$

$\left(\frac{d}{2} - \delta\right)^2 + h_5{}^2 = f^2$

$h_5{}^2 = f^2 - \frac{d^2}{4} + \delta d - \delta^2$ [14]

Considering triangle **CDK**

$\overline{CK} = \left(\frac{d}{2} + \delta\right), \overline{DK} = h_5, \overline{CD} = b$

$\overline{CK}^2 + \overline{DK}^2 = \overline{CD}^2$

$\left(\frac{d}{2} + \delta\right)^2 + h_5{}^2 = b^2$

$h_5{}^2 = b^2 - \frac{d^2}{4} - \delta d - \delta^2$ [15]

Equating [14] and [15]

$f^2 - \frac{d^2}{4} + \delta d - \delta^2 = b^2 - \frac{d^2}{4} - \delta d - \delta^2$

$2\delta d = b^2 - f^2$ [16]

Considering triangle **BCE**

$\overline{CE} = h_6 , \overline{BE} = \left(\frac{f}{2} - \emptyset\right), \overline{BC} = d$

$\overline{CE}^2 + \overline{BE}^2 = \overline{BC}^2$

$h_6{}^2 + \left(\frac{f}{2} - \emptyset\right)^2 = d^2$

$h_6{}^2 = d^2 - \frac{f^2}{4} + \emptyset f - \emptyset^2$ [17]

Considering triangle **CDE**

$\overline{CE} = h_6$, $\overline{DE} = \left(\frac{f}{2} + \emptyset\right), \overline{CD} = b$

$\overline{CE}^2 + \overline{DE}^2 = \overline{CD}^2$

$h_6{}^2 + \left(\frac{f}{2} + \emptyset\right)^2 = b^2$

$h_6{}^2 = b^2 - \frac{f^2}{4} - \emptyset f - \emptyset^2$ [**18**]

Equating [**17**] and [**18**]

$d^2 - \frac{f^2}{4} + \emptyset f - \emptyset^2 = b^2 - \frac{f^2}{4} - \emptyset f - \emptyset^2$

$2\emptyset f = b^2 - d^2$ [**19**]

Adding [**13**], [**16**] and [**19**]

$2\beta b + 2\delta d + 2\emptyset f = d^2 - f^2 + b^2 - f^2 + b^2 - d^2$

$2\beta b + 2\delta d + 2\emptyset f = 2(b^2 - f^2)$

$\beta b + \delta d + \emptyset f = b^2 - f^2$

But $b^2 - f^2 = 2\delta d$

$\beta b + \delta d + \emptyset f = 2\delta d$

$\therefore\ \beta b + \emptyset f = \delta d$ [**20**]

Subtracting [**10**] from [**20**]

$\beta b - \alpha a = \delta d - \gamma c$

$\therefore\ \alpha a + \delta d = \beta b + \gamma c$ **(Q.E.D)**

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