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Approximation of solution of Hammerstein type of equation with strongly accretive bounded nonlinear operator mappings in real Banach space

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Abstract

Let E be a real Banach space with $q-$ uniformly smooth. Suppose that the mapping $F, K : E \to E$ be strongly accretive with $D(K) = R(F) = E$. Suppose that the Hammertein equation $u + KFu = 0$ has a solution in E. We defined here a new algorithm and prove strong convergency of a unique solution to the Hammerstein equation u^*. Further more our technique of prove is of independent of interest.

AMS subject classification:
Keywords: Strongly accretive Mapping, Hammerstein Equations, Iterative Methods.

1. Introduction

Let H be a real Hilbert space. A mapping $A : H \to H$ is said to be monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for every $x, y \in D(A)$ is called monotone and strongly monotone if there exist $k \in (0, 1)$ and for every $x, y \in D(A)$ then $\langle Ax - Ay, x - y\rangle \geq k\|x - y\|$.

Such operators have been studied extensively (see, e.g., Bruck Jr [21], Chidume [10], Martinet [2], Reich [24], Rockafellar [22]) because of their role in convex analysis, in certain partial differential equations, in nonlinear analysis and in optimization theory.

The extension of the *monotonicity* definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. They appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions. (see, e.g., Pascali and Sburian [16], p. 101, Rockafellar [22]).

The *first extension* involves mapping A from E to E^*. Here and in the sequel, $\langle \cdot, \cdot \rangle$ stands for the duality pairing between (a possible normed linear space) E and its dual E^*. A mapping $A : E \to E^*$ with domain $D(A)$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0. \tag{1.1}$$

A is called *strongly monotone* if there exists $k \in (0, 1)$ such that for each $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle \geq k\|x - y\|^2. \tag{1.2}$$

The *second extension* of the notion of monotonicity involves mapping A from E *into itself*. Let E be a real normed space, E^* its dual space. The map $J : E \to 2^{E^*}$ defined by:

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|, \ \|x^*\| = \|x\|\}$$

is called the *normalized duality map* on E. A mapping $A : E \to E$ with domain $D(A)$ is called *accretive* if for all $x, y \in D(A)$, the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \ \ \forall s > 0. \tag{1.3}$$

As a consequence of a result of Kato [25], it follows that A is accretive if and only if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.4}$$

Finally, A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2. \tag{1.5}$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

The accretive operators were introduced independently in 1967 by Browder [20] and Kato [25]. Interest in such mappings stems mainly from their firm connection with equation of evolution. It is known that many physical significant problems can be transformed into a initial value problems of the form

$$\frac{dy}{dx} + Ax(t) = 0 \qquad x(0) = x_0 \tag{1.6}$$

where A is an accretive operator in an appropriate Banach space. Typical examples, where such evolution equations occur, can be found in the heat, wave or Schrodinger equations. If in (1.6), $x(t)$ is independent of t, then (1.6) reduces to

$$Au = 0 \tag{1.7}$$

whose solutions correspond to the equilibrium points of system (1.6). Consequently, considerable research efforts have been devoted, especially within the past twenty years or so, to methods of finding approximate solutions (if exist) of (1.7), and hence,

$$u + Au = 0. \tag{1.8}$$

One important generalization of (1.8) is the so-called equation of Hammerstein type (see, e.g., [1]) where a nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_{\Omega} k(x, y) f(y, u(y)) dy = h(x) \tag{1.9}$$

where dy is a $\sigma-$finite measure on the measure space Ω. The real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear, and h is a given function on Ω. Now if we define an operator K by

$$Kv(x) := \int_{\Omega} k(x, y) v(y)) dy \qquad x \in \Omega \tag{1.10}$$

and the so-called superposition or Nemytskii operator by $Fu(y) := f(y, u(y))$, then the integral equation (1.9) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.11}$$

where, without loss of generality, we have taken $h = 0$.

For the iterative approximation of solutions of (1.7) and (1.8), the monotonicity/accretivity of A is crucial. The Mann iteration scheme (see, e.g., Mann [27] has successfully been employed (see, e.g., the recent monographs of Berinde [26] and Chidume [3]. The recurrence formulas used involved K^{-1} which is also assumed to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not to be monotone. In the special case in which the operators are defined on subsets D of E which are compact, Brezis and Browder [18] proved the strong convergence of a suitably defined Galerkin approximation to a solution of (1.11) (see also Brezis and Browder [19]. In fact, they proved the following Theorem.

Theorem 1.1. (Brezis and Browder) Let H be a separable real Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation

$$(I + KF)u = f, \tag{1.12}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.13}$$

where $K_n = P_n^* K P_n : H \to C_n$ and $F_n = P_n F P_n^* : C_n \to H$, where the symbols have their usual meanings (see, [16]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.13) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of (1.12).

Theorem 1.1 is a special case of the actual theorem of Brezis and Browder in which the Banach space is a separable real Hilbert space. The main theorem of Brezis and Browder is proved in an arbitrary separable real Banach space.

We observe that the Galerkin method of Brezis and Browder is not iterative. The first attempt to construct an iterative method for the approximation of a solution of a Hammerstein equation, as far as we know, was made by Chidume and Zegeye [4] who constructed a sequence in the cartesian product $E \times E$ and proved the convergence of the sequence to a solution of the Hammerstein equation. In subsequent papers, [5], [6] , these authors were able to construct explicit coupled algorithms in the original space E which converge strongly to a solution of the equation. Following this, Chidume and Djitte studied this explicit coupled algorithms and proved several strong convergence theorems see, [7],[8] and [9].

In 2005, Chidume and Zegeye [8] constructed an iterative process as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - w) \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - w) \quad n \in \mathbb{N} \end{aligned} \tag{1.14}$$

where H is a real Hilbert space, F and $K : H \to H$ are bounded monotone mappings satisfying the range condition, $w \in H$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are sequences in $(0, 1)$. Chidume and Zegeye [11] show that this sequence converges strongly to the solution of (1.11) under suitable conditions.

In 2011, Chidume and Ofoedu [19] introduced a coupled explicit iterative process as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1) \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1) \quad n \in \mathbb{N} \end{aligned} \tag{1.15}$$

where E is a uniformly smooth real Banach space, F and $K : E \to E$ are bounded and monotone mappings, and $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are sequences in $(0, 1)$). Chidume and Ofoedu [19] gave a strong convergence theorem for approximation of the solution of (1.11) under suitable conditions.

In 2012, Chidume and Djitte [13] and again Djitte and M. Sene [17] came up with the following iterative scheme respectively:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1) \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1) \quad n \in \mathbb{N} \end{aligned} \tag{1.16}$$

where H is a real Hilbert space, F and K are bounded and maximal monotone mappings, $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are sequences in $(0, 1)$) and Chidume and Djitte [13] show that this iterative process converges to an approximate solution of nonlinear equations of Hammerstein type equation (1.11) under suitable conditions.

Theorem 1.2. [Djitte and M. Sene] [17] Let E be a real Banach space and $F, K : E \to E$be Lipschitz and accretive maps with $D(K) = R(F) = E$ and Lipschitz constants L_1 and L_2, respectively. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively from arbitrary points $u_1, v_1 \in E$ as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1) \quad n \geq 1 \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1) \quad n \geq 1 \end{aligned} \tag{1.17}$$

where $\{\lambda_n\}, \{\alpha_n\}$, and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the certain conditions:

They also proved solution of the hammerstein equation in reflexive real Banach space containing all L_p space $1 < p < \infty$.

In 2013, Chih-Sheng Chuang [15]. Consider the following iterative scheme

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1) \\ v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1) \quad n \in \mathbb{N} \end{aligned} \tag{1.18}$$

where H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are sequences in $(0, 1)$), and he proved a strong convergence of a solution to $u + KFu = 0$.

Motivated by the previous works, in this paper, it is our purpose to construct and iterative scheme in real Banach space that will converges strongly to a unique solution of the Hammerstein equation (if the solution exist) where the operators K and F are strongly accretive. We use an explicit iterative process and furthermore, our results improve related results in the literature.

2. Preliminaries

Theorem 2.1. (Xu , [23]) Let E be a real and $q-$ uniformly smooth Banach space then there exist a constant $d_q > 0$ such that for all $x, y \in E$,

$$\|x + y\| \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q, \tag{2.1}$$

Lemma 2.2. [14] Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Then for all $x, y \in E$, the following inequality holds

$$\|x + y\|^q + \|x - y\|^q \geq \frac{2^q}{1 + c_q}\left(\|x\|^q + \|y\|^q\right)$$

Lemma 2.3. Assume that E is a real $q-$uniformly smooth Banach space, $q > 1$. Let x_1, x_2, y_1 and y_2 be points in E. Then the following estimates holds.

$$\begin{aligned} A(x_1, x_2, y_1, y_2) &= \langle y_1 - y_2, j(x_1 - x_2)\rangle - \langle x_1 - x_2, j(y_1 - y_2)\rangle \\ &\leq \frac{(1 + c_q)(1 + d_q) - 2^q}{q(1 + c_q)}\left(\|x_1 - x_2\|^q + \|y_1 - y_2\|^q\right) \end{aligned}$$

Proof. Since E is real $q-$uniformly smooth the for every $x, y \in E$, the following inequality holds

$$\langle y, j_q(x)\rangle \leq \frac{1}{q}\left(\|x\|_E^q + \|y\|_E^q - \|x - y\|_E^q\right) \tag{2.2}$$

Replacing x by $x_1 - x_2$ and y by $y_1 - y_2$ in inequality (2.2), gives

$$\langle y_1 - y_2, j_q(x_1 - x_2)\rangle \leq \frac{1}{q}\left(\|x_1 - x_2\|_E^q + \|y_1 - y_2\|_E^q - \|(x_1 - x_2) - (y_1 - y_2)\|_E^q\right) \tag{2.3}$$

Interchanging x and y and replacing x by $x_1 - x_2$ and y by $y_1 - y_2$ in inequality (2.2), gives

$$\langle x_1 - x_2, j_q(y_1 - y_2)\rangle \leq \frac{1}{q}\left(\|y_1 - y_2\|_E^q + \|x_1 - x_2\|_E^q - \|(x_1 - x_2) + (y_1 - y_2)\|_E^q\right) \tag{2.4}$$

Combining (2.3) and (2.4) with lemma 2.2, gives

$$\begin{aligned} A(x_1, x_2, y_1, y_2) &= \langle y_1 - y_2, j(x_1 - x_2)\rangle - \langle x_1 - x_2, j(y_1 - y_2)\rangle \\ &\leq \frac{(1 + c_q)(1 + d_q) - 2^q}{q(1 + c_q)}\left(\|x_1 - x_2\|^q + \|y_1 - y_2\|^q\right) \end{aligned}$$

and this complete the proof. ■

3. Main Results

Theorem 3.1. Let E be a real Banach space with $q-$ uniformly smooth. Suppose that the mapping $F, K : E \to E$ be strongly accretive with $D(K) = D(F) = E$. Suppose that the Hammertein equation $u + KFu = 0$ has a solution in E. Let $\{\lambda_n\}_{n=1}$ and

$\{\theta_n\}_{n=1}$ be sequences in $(0,1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively for arbitrary $u_1, v_1 \in E$ by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1) \quad n \geq 1 \\ v_{n+1} &= v_n - \lambda_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1) \quad n \geq 1 \end{aligned} \tag{3.1}$$

where $\{\lambda_n\}_{n=1}$ and $\{\theta_n\}_{n=1}$ satisfies the following conditions

1. $\lim_{n\to\infty} \theta_n = 0$
2. $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty \qquad \lambda_n = o(\theta_n)$
3. $\lim_{n\to\infty} \left(\frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n} \right) = 0$

then the sequence $\{u_n\}$ converges to u^*, a solution to $u + KFu = 0$ and the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Proof. Let $w = E \times E$ with a norm $\|w\| = \left(\|u\|_E^q + \|v\|_E^q\right)$, where $w = (u, v) \in E \times E$. Let

$$d_0 := min \left\{1, \frac{1}{\Gamma}\right\} \qquad \Gamma = \frac{M^q}{q\lambda(\psi - \theta_n)r^q}$$

Since F and K are bounded, it follows that

$$M_1^q := sup \left\{\|Fu_n - v_n + \theta_n(u^* - u_1)\|\right\} + 1 < \infty, \quad \theta \in (0,1)$$

$$M_2^q := sup \left\{\|Kv_n + u_n + \theta_n(v^* - v_1)\|\right\} + 1 < \infty, \quad \theta \in (0,1)$$

$$M^q =:= M_1^q + M_2^q \qquad \psi = \psi_1 + \psi_2$$

Let $u^* \in E$ and a solution to the Hammerstein equation $u + KFu = 0$ and let u^* be fixed. Let $v^* = Fu^*$ and $w^* = (u^*, v^*)$. Then it is observed that $u^* = -Kv^*$.

Step one. We first prove that $\{w_n\}$ is bounded. By induction, we show that $\|w_n - w^*\| \leq r$ for all $n \geq 1$. Indeed by construction, we have $\|w_1 - w^*\| \leq r$. Suppose that $\|w_n - w^*\| \leq r$ for some $n \geq 1$. Then we will show that $\|w_{n+1} - w^*\| \leq r$ for some $n \geq 1$
Computing as follows

$$\|w_{n+1} - w^*\| \leq \|u_{n+1} - u^*\|_E^q + \|v_{n+1} - v^*\|_E^q$$

Using theorem 2.1

$$\begin{aligned} \|u_{n+1} - u^*\|_E^q &= \|u_n - u^* - \lambda_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1)\|_E^q \\ &\leq (1 - \lambda_n\theta_n) \|u_n - u^*\|_E^q - q_n\lambda_n\langle Fu_n - v_n \\ &\quad +\theta_n \left(u^* - u_1\right), j(u_n - u^*)\rangle + d_q\lambda_n\|Fu_n - v_n + \theta_n \left(u^* - u_1\right)\|_E^q \end{aligned} \tag{3.2}$$

Using the fact that

$$\langle Fu_n - v_n, j(u_n - u^*\rangle = \langle Fu_n - Fu^*, j(u_n - u^*\rangle + \langle v^* - v_n, j(u_n - u^*)\rangle \qquad (3.3)$$

and

$$\theta_n \left(u^* - u_1\right), j(u_n - u^*)\rangle = \theta_n \langle u_n - u_1, j(u_n - u^*)\rangle - \theta_n \langle u_n - u^*, j(u_n - u^*)\rangle \quad (3.4)$$

Putting (3.3) and (3.4) in (3.2), gives the following estimates

$$\begin{aligned} \|u_{n+1} - u^*\|_E^q \;\leq\; & (1 - \lambda_n\theta_n) \|u_n - u^*\|_E^q - q_n\lambda_n \langle Fu_n - Fu^*, j(u_n - u^*)\rangle \\ & -q\lambda_n \langle v^* - v_n, j(u_n - u^*)\rangle \\ & -q\lambda_n\theta_n \langle u_n - u_1, j(u_n - u^*)\rangle + q\lambda_n\theta_n \langle u_n - u^*, j(u_n - u^*)\rangle \\ & +d_q\lambda_n^q M_1^q \end{aligned} \qquad (3.5)$$

Knowing that F is strongly accretive, we have the following estimate

$$\begin{aligned} \|u_{n+1} - u^*\|_E^q \;\leq\; & (1 - \lambda_n\theta_n) \|u_n - u^*\|_E^q - q_n\lambda_n\psi_1 \|u_n - u^*\|_E^q \\ & -q\lambda_n \langle v^* - v_n, j(u_n - u^*)\rangle \\ & -q\lambda_n\theta_n \langle u_n - u_1, j(u_n - u^*)\rangle + q\lambda_n\theta_n \|u_n - u^*\|_E^q \\ & +d_q\lambda_n^q M_1^q \end{aligned} \qquad (3.6)$$

Following the same argument, we also obtain

$$\begin{aligned} \|v_{n+1} - v^*\|_E^q \;=\; & \|v_n - v^* - \lambda_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1) \|_E^q \\ \leq\; & (1 - \lambda_n\theta_n) \|v_n - v^*\|_E^q - q_n\lambda_n \langle Kv_n + u_n \\ & +\theta_n \left(v^* - v_1\right), j(v_n - v^*)\rangle + d_q\lambda_n \|Kv_n + u_n + \theta_n \left(v^* - v_1\right) \|_E^q \end{aligned} \qquad (3.7)$$

Using the fact that

$$\langle Kv_n + u_n, j(v_n - v^*\rangle = \langle Kv_n - Kv^*, j(v_n - v^*)\rangle + \langle u_n - u^*, j(v_n - v^*)\rangle \qquad (3.8)$$

and

$$\theta_n \left(v^* - v_1\right), j(v_n - v^*)\rangle = \theta_n \langle v_n - v_1, j(v_n - v^*)\rangle - \theta_n \langle v_n - v^*, j(v_n - v^*)\rangle \quad (3.9)$$

Putting (3.8) and (3.9) in (3.7), gives the following estimates

$$\begin{aligned} \|v_{n+1} - v^*\|_E^q \;\leq\; & (1 - \lambda_n\theta_n) \|v_n - v^*\|_E^q - q_n\lambda_n \langle Kv_n - Kv^*, j(v_n - v^*)\rangle \\ & -q\lambda_n \langle u_n - u^*, j(v_n - v^*)\rangle \\ & -q\lambda_n\theta_n \langle v_n - v_1, j(v_n - v^*)\rangle + q\lambda_n\theta_n \langle v_n - v^*, j(v_n - v^*)\rangle \\ & +d_q\lambda_n^q M_2^q \end{aligned} \qquad (3.10)$$

Knowing that K is strongly accretive, we have the following estimate

$$\begin{aligned}\|v_{n+1}-v^*\|_E^q &\leq (1-\lambda_n\theta_n)\,\|v_n-v^*\|_E^q - q_n\lambda_n\psi_2\|v_n-v^*\|_E^q \\ &\quad -q\lambda_n\langle u_n-u^*, j(v_n-v^*)\rangle \\ &\quad -q\lambda_n\theta_n\langle v_n-v_1, j(v_n-v^*)\rangle + q\lambda_n\theta_n\|v_n-v^*\|_E^q \\ &\quad +d_q\lambda_n^q M_2^q \qquad (3.11)\end{aligned}$$

Combining (3.6) and (3.11) gives the following estimate

$$\begin{aligned}\|w_{n+1}-w^*\| &\leq (1-\lambda_n\theta_n - q\lambda_n\psi + q\lambda_n\theta_n)\,\|w_n-w^*\| + d_q\lambda_n^q M^q \\ &- q\lambda_n\left(\langle u_n-u^*, j(v_n-v^*)\rangle - \langle v_n-v^*, j(u_n-u^*)\rangle\right) \\ &- q\lambda_n\theta_n\left(\langle u_n-u_1, j(u_n-u^*)\rangle + \langle v_n-v_1, j(v_n-v^*)\rangle\right) \qquad (3.12)\end{aligned}$$

$$\begin{aligned}\|w_{n+1}-w^*\| &\leq (1-\lambda_n\theta_n - q\lambda_n\psi + q\lambda_n\theta_n)\,\|w_n-w^*\| + d_q\lambda_n^q M^q \\ &- q\lambda_n A \\ &- q\lambda_n\theta_n\left(\langle u_n-u_1, j(u_n-u^*)\rangle + \langle v_n-v_1, j(v_n-v^*)\rangle\right) \qquad (3.13)\end{aligned}$$

where $\langle u_n-u^*, j(v_n-v^*)\rangle - \langle v_n-v^*, j(u_n-u^*)\rangle = A$
Using lemma 2.3, we estimate

$$A = \frac{\gamma}{q}\left(\|w_n-w^*\|\right) \qquad (3.14)$$

where $\dfrac{\gamma}{q} = \dfrac{(1+c_q)(1+d_q)-2^q}{q(1+c_q)}$. Using Schwartz and Minkowski inequalities and for the fact that $\|j_q(x)\| = \|x\|^{q-1}$, we obtain

$$\langle u_n-u_1, j(u_n-u^*)\rangle \leq \|u_n-u_1\|\|u_n-u^*\|^{p-1} \qquad (3.15)$$

which implies

$$\langle u_n-u_1, j(u_n-u^*)\rangle \leq \frac{1}{q}\|u_n-u_1\| + \frac{1}{q'}\|u_n-u^*\| \qquad (3.16)$$

and also

$$\langle v_n-v_1, j(v_n-v^*)\rangle \leq \|v_n-v_1\|\|v_n-v^*\|^{p-1} \qquad (3.17)$$

which implies

$$\langle v_n-v_1, j(v_n-v^*)\rangle \leq \frac{1}{q}\|v_n-v_1\| + \frac{1}{q'}\|v_n-v^*\| \qquad (3.18)$$

where q' is the Holder's conjugate of q
Putting (3.14), (3.16) and (3.18) in (3.13), gives

$$\begin{aligned}
\|w_{n+1} - w^*\| &\leq (1 - \lambda_n\theta_n - q\lambda_n\psi + q\lambda_n\theta_n)\|w_n - w^*\| + d_q\lambda_n^q M^q - q\lambda_n A \\
&- \frac{q\lambda_n\theta_n}{q}\|w_n - w_1\| - \frac{q\lambda_n\theta_n}{q'}\|w_n - w^*\| \\
&\leq (1 - \lambda_n\theta_n - q\lambda_n\psi + q\lambda_n\theta_n)\|w_n - w^*\| + d_q\lambda_n^q M^q - q\lambda_n A \\
&- \lambda_n\theta_n\|w_n - w_1\| - (q-1)\lambda_n\theta_n\|w_n - w^*\| \\
&\leq (1 - q\lambda_n\theta_n - q\lambda_n(\psi - \theta_n))\|w_n - w^*\| + d_q\lambda_n^q M^q - q\lambda_n A \\
&- \lambda_n\theta_n\|w_n - w_1\| \qquad (3.19)
\end{aligned}$$

Now taking the definition of d_0 and setting

$$\|w_n - w_1\| \leq 2r^q$$

$$\begin{aligned}
\|w_{n+1} - w^*\| &\leq (1 - q\lambda_n\theta_n)r^q - 2\lambda_n\theta_n r^q - q\lambda_n A \\
&\leq (1 - q\lambda_n\theta_n)r^q - 2\lambda_n\theta_n r^q - \lambda_n\gamma_n r^q \\
&\leq (1 - \lambda_n(q\theta_n - 2\theta_n - \gamma_n))r^q \qquad (3.20)
\end{aligned}$$

Therefore since $\|w_{n+1} - w^*\| \leq r^q$ then the sequence $\{w_n\}$ is bounded.

Step Two We now show that $\{w_n\}$ is a Cauchy sequence in E. From the same computation we have

$$\|w_{n+1} - w^*\| \leq (1 - \lambda_n(q\theta_n - 2\theta_n - \gamma_n))r^q$$

Since $\lambda_n(q\theta_n - 2\theta_n - \gamma_n) \in (0, 1)$ and hence it follows that $\{w_n\}$ is Cauchy in E and so it converges to (u^*, v^*) satisfying $v^* = Fu^*$ and u^* is the unique solution to the Hammerstein equation $u^* + KFu^* = 0$. This completes the proof. ■

Corollary 3.2. Let E be a real Banach space with $q-$ uniformly smooth. Suppose that the mapping $F, K : E \to E$ be strongly accretive with $D(K) = D(F) = E$. Suppose that the Hammertein equation $u + KFu = 0$ has a solution in E. Let $\{\lambda_n\}_{n=1}$ and $\{\theta_n\}_{n=1}$ be sequences in $(0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively for arbitrary $u_1, v_1 \in E$ by

$$\begin{aligned}
u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1) \quad n \geq 1 \\
v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1) \quad n \geq 1 \qquad (3.21)
\end{aligned}$$

then the sequences $\{u_n\}$ converges strongly to a unique solution u^* of the Hammerstein equation $u + KFu = 0$ with $v^* = Fu^*$.

Corollary 3.3. Suppose $E = L_p(2 \leq p < \infty)$. Let $F, K : E \to E$ be strongly accretive map with $D(K) = R(F) = E$. Suppose that the Hammertein equation $u +$

$KFu = 0$ has a solution in E. Let $\{\lambda_n\}_{n=1}$ and $\{\theta_n\}_{n=1}$ be sequences in $(0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively for arbitrary $u_1, v_1 \in E$ by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1) \quad n \geq 1 \\ v_{n+1} &= v_n - \lambda_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1) \quad n \geq 1 \end{aligned} \tag{3.22}$$

converges to $(u^*, v^*) \in E$ and u^* is the unique solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. Since $L_p, 2 \leq p < \infty$ are $q-$ uniformly smooth, then the prove follows from (3.1). ■

Corollary 3.4. Let E be a real Banach space and $F, K : E \to E$ be strongly accretive map with $D(K) = R(F) = E$. Suppose that the Hammertein equation $u + KFu = 0$ has a solution in E. Let $\{\lambda_n\}_{n=1}$ and $\{\theta_n\}_{n=1}$ be sequences in $(0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively for arbitrary $u_1, v_1 \in E$ by

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1) \quad n \geq 1 \\ v_{n+1} &= v_n - \lambda_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1) \quad n \geq 1 \end{aligned} \tag{3.23}$$

converges to $(u^*, v^*) \in E$ and u^* is the unique solution of $u + KFu = 0$ with $v^* = Fu^*$.

Remark 3.5. Real sequences that satisfy conditions $(i) - (iii)$ are $\lambda_n = (n + 1)^{-a}$ and $\theta_n = (n + 1)^{-b}, n = 1$ with $0 < b < a, \frac{1}{2} < a < 1$ and $a + b < 1$.

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