A Study on Some Curvature Properties of Almost $C(\lambda)$ Manifold

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Abstract

The plan of the present paper is to study some curvature properties of almost $c(\lambda)$ manifolds. We shall consider Einstein semi symmetric almost $c(\lambda)$ manifolds and such manifolds satisfying $E.S = 0$, $S.E = 0$, $E.R = 0$, where $E$ is the Einstein tensor, $R$ is the Riemannian curvature tensor, $S$ is the Ricci tensor of the manifold. We shall also consider $\phi$-Ricci symmetric almost $c(\lambda)$ manifolds.

Keywords and Phrases: Almost contact manifolds, almost $c(\lambda)$ manifolds, Einstein tensor, $\phi$-Ricci symmetry, Riemannian curvature tensor.


SECTION -1
INTRODUCTION

The notion of almost $c(\lambda)$ manifolds was first given by Janssen and Vanhecke [6]. Again in the paper [7] conformally flat almost $c(\lambda)$ manifolds have been studied. Recently in the papers [1], [2], [3], A. Akbar has studied some curvature properties of almost $c(\lambda)$ manifolds. The notion of Einstein tensor has been introduced to study curvature properties in the paper [3]. $\phi$-Ricci symmetric sasakian manifolds have been studied in the paper [5].

In this paper we would like to study some curvature properties of almost $c(\lambda)$ manifolds. The present paper is organised as follows: we give some preliminary formulae in section-2. In section-3 we study Einstein semisymmetric almost $c(\lambda)$ manifolds. Section 4 contains the study of almost $c(\lambda)$ manifolds satisfying $E.R = 0$, where $E$ is the Einstein tensor, $R$ is the Riemannian curvature tensor, $S$ is the Ricci curvature tensor of the manifolds. The last section contains the study of $\phi$-Ricci symmetric almost $c(\lambda)$ manifolds.
SECTION- 2
Preliminaries : An odd dimensional differentiable manifold is called almost contact manifolds if there exist a 1-1 tensor \( \eta \), a vector field \( X \), and a Riemannian metric \( g \) such that [4].

\[
\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = 1 \tag{2.1}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}
\]

\[
\phi(\xi) = 0, \quad \eta(\phi(X)) = 0 \tag{2.3}
\]

Here \( X, Y \) are differentiable vector fields defined on the manifolds. An almost contact manifolds is called an almost c(\( \lambda \)) manifolds if its curvature tensor \( R \) is given by [7]

\[
R(X, Y)\xi = R(\phi X, \phi Y)\xi - \lambda[g(X, Y) - \phi Xg(X, Y) + g(X, Y)\phi Y] \tag{2.4}
\]

From above equation we also have

\[
R(X,Y)\xi = R(\phi X, \phi Y)\xi - \lambda[|Xg(X, Y)\phi X - g(X, Y)\phi Y]\tag{2.5}
\]

\[
R(\xi, Y)Z = -\lambda[g(Y,Z)\xi - \eta(Z)Y] \tag{2.6}
\]

\[
R(\xi, Y)\xi = -\lambda[\eta(Y)\xi - Y] \tag{2.7}
\]

\[
R(\xi,\xi)Z = 0 \tag{2.8}
\]

The Ricci tensor of almost c(\( \lambda \)) manifolds was deduced in the paper [1]. The Ricci tensor is given below

\[
S(X,Y) = -\lambda[(2n-1)g(X,Y) + \eta(X)\eta(Y)] \tag{2.9}
\]

Here the dimension of the manifold is considered \( 2n+1 \) from above we get the Ricci operator \( Q \) as follows

\[
QX = \lambda (1-2n)X - \lambda \eta(X)\xi \tag{2.10}
\]

The Einstein tensor of a manifold is defined as

\[
E(X,Y) = S(X,Y) - \frac{r}{2}g(X,Y) \tag{2.11}
\]

Where \( S \) is the Ricci curvature tensor and \( r \) is the scaler curvature tensor of the manifold.
SECTION- 3

Definition 3.1

An almost c(λ) manifold will be called Einstein semisymmetric if it satisfies
R(X,Y)Z .E(U,V) = 0. -------------------------------------(3.1)

Let us consider an almost c(λ) manifolds is Einstein semisymmetric then
R(X,Y)Z.E(U,V) = 0.

Now from (2.11) E(U,V) = S(U,V) – r/2g(U,V).

Again using we have E(U,V) = -λ
[(2n-1)g(U,V) +η(U)η(V)] - r/2g(U,V). ----------------(3.2) Now
R(X,Y)Z.E(U,V) =0, means
E(R(X,Y)Z,U) +E(Z,R(X,Y)U) = 0.

Using (2.11) in the above equation we get
S(R(X,Y)Z,U) – r/2g(R(X,Y)Z,U) +
S(Z,R(X,Y)U) – r/2g(Z,R(X,Y)U) = 0.

Putting Z= ξ  in the above equation we get
S(R(X,Y)ξ,U) – r/2g(R(X,Y)ξ,U) +
S (ξ, R(X,Y)U)  – r/2g(ξ,R(X,Y)U) = 0.

Using( 2.5) and (2.9) in the above equation we get,
S(R(ϕ X, ϕ Y)ξ -λ[η(Y)X –η(X)Y] , U) – r/2g(R(ϕ X, ϕ Y)ξ,U)+
λr/2 η(Y)g (X,U)–
λr/2 η(X)g(Y,U)– 2nλ η(R(ξ,Y)U) =0,
S(R(ϕ X, ϕ Y)ξ -λ[η(Y)X –η(X)Y] , U) – r/2g(R(ϕ X, ϕ Y)ξ,U)+
λr/2 η(Y)g (X,U)-
λr/2η(Y)g (X,U) – η(ϕ Y,ϕ U) =0.

Putting X=ξ ,We have
S(-η(ϕ Y,ϕ U) +λr/2η(Y)η(U)- 2nλ η(R(ξ,Y)U)-r/2 η(R(ξ,Y)U)=0,

λ^2[(2n-1)g(ϕ Y,ϕ U) -2λ[ g(Y,U)- η(Y)η(U)] ( 2nλ+r/2) η(ϕ Y,ϕ U)=0,
λ^2(2n-1)g(ϕ Y,ϕ U)- λr/2[ g(Y,U)- η(Y)η(U)] ( 2nλ+r/2) η(ϕ Y,ϕ U)=0,
λ^2[(2n-1)g(ϕ Y,ϕ U) -2λ[ g(Y,U)- η(Y)η(U)] ( 2nλ+r/2) η(ϕ Y,ϕ U)=0,
\(-\lambda^2(2n-1) \cdot g(\phi Y, \phi U) + 2\lambda^2 n \cdot g(\phi Y, \phi U) = 0, \)
\(\lambda^2 g(\phi Y, \phi U) = 0, \) this implies \(\lambda = 0.\) As \(g(\phi Y, \phi U)\) is not zero.

Hence we can state the following theorem.

**Theorem 3.1**

If an almost c(\(\lambda\)) manifold is Einstein semisymmetric, then \(\lambda\) is necessarily zero.

But the converse may not be true always.

**SECTION-4**

In this section we like to study almost c(\(\lambda\)) manifold satisfying \(E.R = 0\) where \(E\) is Einstein tensor and \(R\) is Riemannian curvature tensor. Let us consider an almost c(\(\lambda\)), manifold satisfying \(E.R = 0.\) now \(E.R = 0\) means, \(E(U, R(X,Y)Z) + E(R(X,Y), V) = 0.\)

\(S(U, R(X,Y)Z) - r/2g(U, R(X,Y)Z) + S(V, R(X,Y)Z) - r/2g(V, R(X,Y)Z) = 0.\)

Putting \(Z = \xi,\) we get, \(S(U, R(X,Y) \xi) - r/2g(U, R(X,Y) \xi) + S(V, R(X,Y) \xi) - r/2g(V, R(X,Y) \xi) = 0,\)

\(-\lambda [(2n-1)g(U, R(X,Y) \xi) + \eta(U) \cdot \eta(R(X,Y) \xi)] - r/2g(R(X,Y) \xi, U) - \lambda [(2n-1)g(R(X,Y) \xi, V) + \eta(V) \cdot \eta(R(X,Y) \xi)] = 0,\)

Putting \(X = \xi,\) we have,

\(-\lambda [(2n-1)g(U, -\lambda(\eta(Y) \xi - Y) + \eta(U) \cdot (-\lambda(\eta(Y) \xi - Y))] - r/2g(U, -\lambda(\eta(Y) \xi - Y)) - \lambda [(2n-1)g(V, -\lambda(\eta(Y) \xi - Y) + \eta(V) \cdot (-\lambda(\eta(Y) \xi - Y))] - r/2g(V, -\lambda(\eta(Y) \xi - Y)) = 0,\)

\(\lambda^2(2n-1) \cdot g(\phi^2 Y, U) + r \cdot \lambda^2 /2g(\phi^2 Y, U) + \lambda^2(2n-1)g(\phi^2 Y, V) + r \cdot \lambda^2 /2g(\phi^2 Y, V) = 0,\)

\((2n \lambda^2 - \lambda^2 + r \cdot \lambda /2)(g(\phi^2 Y, U) + g(\phi^2 Y, V)) = 0.\) Finally we get \(\lambda = 0.\)

Thus we are in a situation to state the following.

**Theorem 4.1**

If an almost c(\(\lambda\)) manifold satisfies \(E.R = 0\) then \(\lambda\) is necessarily zero.

But the converse may not be true always.
SECTION -5

ϕ -Ricci symmetric almost c(λ) manifold:

The notion of ϕ- Ricci symmetric sasakian manifolds was introduced by U.C. De and A. Sarkar in the paper [5] following this paper in this section we study ϕ -Ricci symmetric almost c(λ) manifolds.

Definition 5.1

An almost c(λ) manifold will be called ϕ- Ricci symmetric if \( \phi^2(\nabla_wQ)X = 0 \).

The manifold will be called Ricci symmetric if the vector field \( W \) and \( X \) are orthogonal to \( \xi \).

Let us consider an almost c(λ) manifold is ϕ-Ricci symmetric.

Now from (2.10) we get \( QX = \lambda(1-2n)X - \lambda\eta(X)\xi \) now

\[
(\nabla_wQ)X = \nabla_w(QX) - Q(\nabla_wX)
\]

\[
= \nabla_w(\lambda(1-2n)X - \lambda\eta(X)\xi) - \lambda(1-2n)\nabla_wX + \lambda\eta(\nabla_wX)\xi
\]

\[
= (1-2n)(\lambda\nabla_wX + X\nabla_w\lambda) - (\nabla_w\lambda)\eta(X)\xi - \lambda(\nabla_w\eta(X))\xi
\]

\[
+ (\nabla_w\xi)\eta(X) \lambda(1-2n)\nabla_wX - \lambda\eta(\nabla_wX)\xi
\]

\[
= (1-2n)\nabla_w\lambda - (\nabla_w\lambda)\eta(X)\xi - \lambda((\nabla_w\eta)X\xi + \eta(\nabla_wX)\xi + (\nabla_w\xi)\eta(X)) + \lambda\eta(\nabla_wX)\xi
\]

Since the manifold is locally ϕ -Ricci symmetric by definition

\[
(1-2n)\nabla_w\lambda - (\nabla_w\lambda)\eta(X)\xi - \lambda((\nabla_w\eta)X\xi + \eta(\nabla_wX)\xi + (\nabla_w\xi)\eta(X)) = 0,
\]

\[
\nabla_w\lambda(X - 2nX - \eta(X)\xi) - \lambda((\nabla_w\eta)X\xi + (\nabla_w\xi)\eta(X)) = 0.
\]

Replacing \( X \) by \( \phi X \)

\[
(\nabla_w\lambda)(QX - 2n\phi X) - \lambda((\nabla_w\eta)(\phi X)\xi = 0.
\]

If \( \lambda \) is constant then \( \lambda \) must be zero. Thus we are in a position to state the following theorem.

Theorem 5.1

There exist no ϕ - Ricci symmetric almost c(λ) manifold with \( \lambda \) as a non zero constant.
REFERENCES


