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Relations of Centralizers on Semiprime Semirings

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Abstract

We define and study the relations of centralizers on semiprime semiring. In this paper, we prove that an additive mapping T of a 2-torsion free semiprime semiring S into itself such that $T(xyx) = xT(y)x$ for all $x, y \in S$, is a Centralizer. We also show that if S contains a multiplicative identity1, then T is a centralizer.

Key words : Semiring, Semiprime Semiring, Centralizer, Jordan centralizer, left(right) centralizer.

Mathematical Subject Classification (2000): 16Y60

1. INTRODUCTION

Semirings arise erupt in various fields of mathematics. Indeed, the first mathematical structure we encounter- the set of natural numbers is a semiring. Historically semirings first appear implicitly in[6] and the algebraic structure of semiring was introduced by H.S. Vandiver in 1934. Later semirings were investigated by numerous researches in their own right, in order to broaden techniques and to generalize results from ring theory or semigroup theory or in connection with some applications (see [10]).

H.E.Bell and W.S.Martindale [1] established the study of Centralizing mapping on semiprime rings. After many mathematicians made good works on centralizers of semiprime rings (see [2], [11]).M.F. Hoque and A.C. Paul [8] studied the centralizers of semiprime Gamma rings. In [5] Chandramouleeswaran and Thiruveni, made good works on derivations of semirings. Joso Vukman[12] proved that if R is a 2-torsion free Semiprime Ring and $T: R \to R$ is an additive mapping such that $T(xyx) = xT(y)x$ for all $x, y \in R$, then T is a centralizer. Motivated by this D. Mary Florence and R. Murugesan [14] studied the notion of Semirings and proved that every Jordan Centralizer of a 2 - torsion free Semiprime Semiring is a centralizer. Also show that in general, every Jordan centralizer is not a centralizer.The purpose of this paper is to obtain the centralizers on semiprime semirings in the sense of Joso Vukman.

2. PRELIMINARIES

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1

A Semiring is a nonempty set S followed with two binary operation '$+$' and '$.$'such that

(1) $(S, +)$*is a commutative monoid with identity element* '0'.

(2) $(S, .)$*is a monoid with identity element 1*

(3) *Multiplication distributes over addition from either side*

That is $a.(b + c) = a.b + a.c$

$(b + c).a = b.a + c.a$

Definition 2.2

A Semiring S is said to be prime if $xSy = 0 \Rightarrow x = 0$ *or* $y = 0$ *for all* $x, y \in S$

Definition 2.3

A Semiring S is said to be semiprime if $xSx = 0 \Rightarrow x = 0$ *for all* $x \in S$

Definition 2.4

A Semiring S is said to be 2-torsion free if $2x = 0 \Rightarrow x = o$ *for all* $x \in S$

Definition 2.5

A Semiring $S$is said to be commutative Semiring, if $xy = yx$ *for all* $x, y \in S$*, then the set*

$Z(S) = \{\ x \in S,\ xy = yx\ \ for\ all\ y \in S\}$ *is called the center of the Semiring* S.

Definition 2.6

For any fixed $a \in S$, *the mapping* $T(x) = ax$ *is a left centralizer and* $T(x) = xa$ *is a right centralizer.*

Definition 2.7

An additive mapping $T: S \to S$ *is a left (Right) centralizer* $if\ T(xy) = T(x)y, \big(T(xy) = xT(y)\big)$ *for all* $x, y \in S$
A centralizer is an additive mapping which is both left and right centralizer.

Definition 2.8

An additive mapping $T: S \to S$*is Jordan left (Right) Centralizer if*

$$T(xx) = T(x)x, (T(xx) = xT(x))\ \ for\ all\ x \in S$$

Every left centralizer is a Jordan left centralizer but the converse is not in general true.

Definition 2.9

An additive mapping $T: S \to S$ *is a Jordan centralizer if* $T(xy + yx) = T(x)y + yT(x)$ *for all* $x, y \in S$
Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Definition 2.10

An additive mapping $D: S \to S$ *is called a derivation if* $D(xy) = D(x)y + xD(y)$ *for all* $x, y \in S$ *and is called a Jordan derivation if* $D(xx) = D(x)x + xD(x)$ *for all* $x \in S$.

Definition 2.11

If S *is a semiring then* $[x, y] = xy + y'x$ *is known as the commutator of* x *and* y
The following are the basic commutator identities:

$$[xy, z] = [x, z]y + x[y, z] and\ [x, yz] = [x, y]z + y[x, z]\ for\ all\ x, y, z \in S$$

According to [15] $for\ all\ a, b \in S$ we have
$(a + b)' = a' + b'$
$(ab)' = a'b = ab'$
$a'' = a$
$a'b = (a'b)' = (ab)'' = ab$
Also the following implication is valid.
$a + b = o\ implies\ a = b'$ and $a + a' = 0$

Lemma :2.12 (Lemma 2.1 in [13])

Let S be a Semiprime Semiring. Suppose that the relation $axb + bxc = 0$ for all $x \in S$ and some $a, b, c \epsilon S$. In this case $(a + c)xb = 0$ for all $x \in S$.

Theorem : 2.13 (Theorem 4.1 in [14])

Every Jordon centralizer of a 2-torsion free semiprime semiring is a centralizer.

3. OUR MAIN RESULT

Theorem:3.1 *Let S be a 2-torsion free Semiprime Semiring and let $T: S \to S$ be an additive mapping.*

Such that $T(xyx) = xT(y)x$ holds for all $x, y \in S$.Then T is a centralizer.
Proof:

(1) we have $T(xyx) = xT(y)x$

After replacing x by $x + z$ in (1) we obtain

$T[(x + z)y(x + z)] = (x + z)T(y)(x + z)$

$T[(xy + zy)(x + z)] = (xT(y) + zT(y))(x + z)$

$T(xyx + xyz + zyx + zyz) = xT(y)x + xT(y)z + zT(y)x + zT(y)z$

$T(xyx) + T(xyz + zyx) + T(zyz)$

$= xT(y)x + xT(y)z + zT(y)x + zT(y)z$

$xT(y)x + T(xyz + zyx) + zT(y)z$

$= xT(y)x + xT(y)z + zT(y)x + zT(y)z$

(2) $T(xyz + zyx) = xT(y)z + zT(y)x$

Replacing y *by* x and z *by* $y$in (2) we get

(3) $T(x^2y + yx^2) = xT(x)y + yT(x)x$

Putting z *by* x^3 in (2) which implies that

(4) $T(xyx^3 + x^3yx) = xT(y)x^3 + x^3T(y)x$

After replacing y *by* xyx in (3) we obtain

(5) $T(x^3yx + xyx^3) = xT(x)xyx + xyxT(x)x$

Substitution for $y = x^2y + yx^2$ in (1) yields

(6) T$(x^3yx + xyx^3)$= $x^2T(x)yx + xyT(x)x^2$

After comparing (5) with (6) we arrive at

$xT(x)xyx + xyxT(x)x = x^2T(x)yx + xyT(x)x^2$

$xT(x)xyx = x^2T(x)yx + xyT(x)x^2 + xyx'T(x)x$

$xT(x)xyx = x^2T(x)yx + xy[T(x), x]x$

Adding $x^2T(x)yx' + xy[T(x), x]x'$ on both sides, we get

$xT(x)xyx + x^2T(x)yx' + xy[T(x), x]x' = 0$

$x[T(x)x + x'T(x)]yx + xy[T(x), x]x' = 0$

$x[T(x), x]yx + xy[T(x), x]x' = 0$

(7) $x[T(x), x]yx + xy[x, T(x)]x$ =0

From the above we assume, $a = x[T(x), x)], x = y, b = x, c = [x, T(x)]x$

Now applying lemma 2.12 follows that $(x[T(x), x] + [x, T(x)]x)yx = 0$

$([x, T(x)]x + x'[x, T(x)])yx = 0$

(8) $[[x, T(x)], x]yx = 0$

Applying y by $y'[T(x), x]$ in the above relation we get

$[[x, T(x)], x]y'[T(x), x]x = 0$

(9) $[[T(x), x], x]y[T(x), x]x = 0$

Right multiplying (8) by $[T(x), x]$ we obtain

$[[x, T(x)], x]yx[T(x), x] = 0$

(10) $[[T(x), x], x]yx'[T(x), x] = 0$

Adding (9) and (10), we get

$[[T(x), x], x]y\,([T(x), x]x + x'[T(x), x]) = 0$

$[[T(x), x], x]y[[T(x), x], x] = 0$

By the semiprimeness of S, implies

(11) $[[T(x), x], x] = 0$

From the above relation replacing x by $x + y$ and using ((11) we see that

$[[T(x + y), x + y], x + y] = 0$

$[[T(x) + T(y), x + y], x + y] = 0$

$[[T(x), x] + [T(x), y] + [T(y), x] + [T(y), y], x + y] = 0$

$$\big[[T(x), x], x\big] + [[T(x), y], x] + [[T(y), x], x] + [[T(y), y], x] + [[T(x), x], y] + [[T(x), y], y] + [[T(y), x], y] + [[T(y), y], y] = 0$$

$\big[[T(x),y],x\big]+\big[[T(y),x],x\big]+\big[[T(x),y],y\big]+\big[[T(y),x],y\big]+$

$[[T(y),y],x]+[[T(x),x],y]=0$

Replacing $x\ by\ x'$ in the above, we get

$\big[[T(x),y],x\big]+\big[[T(y),x],x\big]+\big[[T(x'),y],y\big]+\big[[T(y),x'],y\big]+$

$[[T(y),y],x']+[[T(x),x],y]=0$

Adding the above two relations, we obtain

$2\big[[T(x),y],x\big]+2\big[[T(y),x],x\big]+2[[T(x),x],y]=0$

(12) $\big[[T(x),y],x\big]+\big[[T(y),x],x\big]+[[T(x),x],y]=0$

Putting xyx for y in (12)and using (12) we obtain

$\big[[T(x),xyx],x\big]+\big[[T(xyx),x],x\big]+[[T(x),x],xyx]=0$

$[xy[T(x),x],x]+[x[T(x),y]x,x]+\big[[T(x),x]yx,x\big]$

$+[x[T(y),x]x,x]+x\big[[T(x),x],y\big]x=0$

$[T(x),x][y,x]x+x[y,x][T(x),x]+x\big[[T(x),y],x\big]x+x[[T(y),x]x$

$+x\big[[T(x),x],y\big]x=0$

$[T(x),x][y,x]x+x[y,x][T(x),x]=0$

$[T(x),x]yx^2+[T(x),x]x'yx+xyx[T(x),x]+x^2y'[T(x),x]=0.$

Replacing $y\ by\ y'$ in the above, we get

$[T(x),x]y'x^2+[T(x),x]xyx+x'yx[T(x),x]+x^2y[T(x),x]=0.$

Application of (7), gives

$[T(x),x]yx^2+x^2y'[T(x),x]=0$

Left multiplication of the above relation by , we get

$x[T(x),x]yx^2+x^3y'[T(x),x]=0$

Using (7), we get

(14) $xy[T(x),x]x^2+x^3y'[T(x),x]=0$

Left multiplying by $T(x)$, we get

(15) $T(x)xy[T(x),x]x^2+T(x)x^3y'[T(x),x]=0$

Replacing $y\ by\ T(x)y$ in (14) we get

$xT(x)y[T(x),x]x^2+x^3T(x)y'[T(x),x]=0$

Replace y by y' in the above, we obtain

(16) $xT(x)y'[T(x),x]x^2 + x^3T(x)y[T(x),x] = 0$

Adding (15) and (16), we get

$T(x)xy[T(x),x]x^2 + T(x)x^3y'[T(x),x] + xT(x)y'[T(x),x]x^2 +$

$x^3T(x)y[T(x),x] = 0$

$[x^3,T(x)]y[T(x),x] + \ [T(x),x]y[T(x),x]x^2 = 0$

Applying lemma 2.12,in the above follows that

$([x^3,T(x)] + [T(x),x]x^2)y[T(x),x] = 0$

$([xxx,T(x)] + [T(x),x)]x^2)y[T(x),x] = 0$

$([x,T(x)]x^2 + x[x,T(x)]x + x^2[x,T(x)] + [T(x),x]x^2)y[T(x),x] = 0$

$(x[x,T(x)]x + x^2[x,T(x)] + [T(x),x]xx' + [T(x),x]xx)y[T(x),x] = 0$

This implies, $(x[x,T(x)]x + x^2[x,T(x)])\ y[T(x),x] = 0$

Replace y by y'in the above, we get

$(x[T(x),x]x + x^2[T(x),x])\ y[T(x),x] = 0$

Using (11) , we obtain

$(x[T(x),x]x + x[T(x),x]x)\ y[T(x),x] = 0$

$2x[T(x),x]xy[T(x),x] = 0$

S is 2-torsion free Semiprime Semiring, gives $x[T(x),x]xy[T(x),x] = 0$

Right multiplication of the above relation by x and replacing $y\ by\ \ yx$ gives

$x[T(x),x]xyx[T(x),x]x = 0$

(17) By the Semiprimeness of S implies, x$[T(x),x]x = 0$

Putting y by yx in(7) and using the above, we get

(18) $x[T(x),x]yx^2 = 0$

The substitution $yT(x)$for y in(18) yields

(19) $x[T(x),x]yT(x)x^2 = 0$

Right multiplication of (18) by $T(x)$ we get

$x[T(x),x]yx^2T(x) = 0$

Replacing $y\ by\ y'$, we get

(20) $x[T(x),x]y'x^2T(x) = 0$

Adding (19) and (20)., we obtain

$x[T(x),x]y(T(x)x^2$+$x'^2T(x)$)=0

$x[T(x),x]y[T(x),x^2]=0$

$x[T(x),x]y[T(x),xx]=0$

$x[T(x),x]y([T(x),x]x+x[T(x),x])=0$

According to (7) one can replace $x[T(x),x]$ by $[T(x),x]x$ which gives

$2x[T(x),x]yx[T(x),x]=0$

Since we are in semiprime semirings we conclude that

(21) $x\,[T(x),x]=0$

According to (11) it is easy to see that

(22) $[T(x),x]x=0$

Linearizing the above result (see how to obtain (12) from(11)) we reach

$[T(x),y]x+[T(y),x]x+[T(x),x]y=0$

Right multiplication of the above relation by $[T(x),x]$and using (21) this reduces to

$[T(x),x]y[T(x),x]=0$

Semiprimeness of S implies,

(23) $[T(x),x]=0$

Next we will intent to prove the result

(24) $T(xy+yx)=T(y)x+xT(y)$

In order to prove the above we need to prove the following relation

(25) $[G(x,y),x]=0\ \ ,x,y\in S$

Where $G(x,y)$ stands for $T(xy+yx)+T(y)x'+x'T(y)$,

First we replacing y by $xy+yx$ in equation (1) gives

(26) $T(x^2yx+xyx^2)=xT(xy+yx)x$

On the other hand putting $z\ by\ x^2$ in (2) we obtain

(27) $T(xyx^2+x^2yx)=xT(y)x^2+x^2T(y)x$

Comparing (26) and (27) it is clear that

(28) $xG(x,y)x=0$

Now let us prove (25). Replacing x by $x+y$ in relation(23)

(29) $[T(x),y]+[T(y),x]=0\quad x,y\in S.$

After replacing $y\ by\ xy + yx$ in the above and using relation (23) leads to

(30) $x[T(x), y] + [T(x), y]x + [T(xy + yx), x] = 0$

using (29) ,the above relation becomes

$[T(xy + yx), x] + x'[T(y), x] + [T(y), x]x' = 0$

This can be written in the form $[T(xy + yx) + T(y)x' + x'T(y), x] = 0$

Therefore (25) is proved.

Substituting x by $x + z$ in (28) and using (28) gives

$xG(x, y)z + zG(x, y)x + zG(x, y)z + xG(z, y)x + xG(z, y)z + zG(z, y)x = 0$

Replacing $x\ by\ x'$ in the above we get

$xG(x, y)z + zG(x, y)x + z'G(x, y)z + xG(z, y)x + x'G(z, y)z +$

$zG(z, y)x' = 0,$ x,y,z∈ S

Adding the above two relations and using S is 2-toesion free, we get

$xG(x, y)z + zG(x, y)x + xG(z, y)x = 0,$ x,y,z∈ S

Right multiplication of the above relation by $G(x, y)x$ gives because of (28)

$xG(x, y)z\, G(x, y)x = 0$

Using (25) , the above relation can be written in the form

$xG(x, y)zxG(x, y) = 0$

By the semiprimeness of S we get

(31) $xG(x, y) = 0$

From (25) and (31) we also get

(32) $G(x, y)x = 0$

The linearization of (32) by putting $x\ by\ x + z$ we get

(33) $G(x, y)z + G(z, y)x = 0$ for all $x, y, z \in S$

Right Multiplication of the above by $G(x, y)$ and applying (31) we arrive at

$G(x, y)zG(x, y) = 0$

By the semiprimeness of S it follows that $G(x, y) = 0$.

So the proof of (24) is completed.

In particular when y by x in relation (24) reduces to

$2T(x^2) = T(x)x + xT(x)\ , x \in S$

Using (23) and S is 2-torsion free Semiprime Semiring, the above relation yields

$T(x^2) = T(x)x, \quad x \in S$

And $T(x^2) = xT(x)$, $x \in S$

This means that T is a Left and Right Jordan centralizer

By Theorem 2.13 follows that $T$is a centralizer . Which completes the proof of the theorem.

Putting $y = x$ in relation (1) we obtain $T(xxx) = xT(x)x$, $x \in S$. The question arises whether in a 2-torsion free Semiprime Semiring the above relation implies that T is a centralizer. Unfortunately, we were unable to answer it affirmative because S has an identity element 1.

Theorem :3.2

Let S be a 2-torsion free Semiprime Semiring with identity element 1and let $T: S \to S$ be an additive mapping. Suppose that $T(xxx) = xT(x)x$ holds for all $x \in S$.Then$T$ is a centralizer.

Proof: BY Hypothesis, we have

(34) $T(xxx) = xT(x)x$

Putting $x = x + 1$ in the above follows that

$$T(x^3) + 3T(x^2) + 3T(x) + T(1)$$
$$= xT(x)x + xT(1)x + T(x)x + T(1)x + xT(x) + xT(1) + T(x) + T(1)$$

Putting $T(1) = a$ and apply (34) in the above relation implies that

(35) $3T(xx) + 2T(x) = xax + T(x)x + ax + xT(x) + xa$

Putting x by x' in the above, we get

$3T(xx) + 2T(x') = xax + T(x)x + ax' + xT(x) + x'a$

Adding the above two relations ,we obtain

(36) $6T(xx) = 2xax + 2T(x)x + 2xT(x)$

Once again substituting $xby\ x + 1\ \ in\ (35)$and a stands for T(1) yields

$3T(x^2) + 8T(x) + 5a = xax + 3xa + 3ax + 5a + T(x)x + xT(x) + 2T(x)$

(37) $3T(x^2) + 6T(x) = xax + 3xa + 3ax + T(x)x + xT(x)$

From (35) we have $3T(xx) = 2T'(x) + xax + T(x)x + ax + xT(x) + xa$

Substituting the above in (37), we get

$4T(x) = 2xa + 2ax$

(38) $2T(x) = xa + ax$

We shall prove that $a \in Z(S)$

Substituting (38) in (36), we obtain

$6T(xx) = 2xax + (ax + xa)x + x(ax + xa)$

(39) $6T(xx) = 4xax + axx + xxa$

From (38) we obtain $6T(xx) = 3xxa + 3axx$

Substituting the above in (39), we get

(40) $3xxa + 3axx = 4xax + axx + xxa$

$2axx + 2xxa = 4xax$

Adding $4xax'$ on both sides, we get

$2axx + 2xxa + 4xax' = 0$

(41) $axx + xxa + 2xax' = 0$

The above relation can be written in the form

(42) $[[a, x], x] = 0$

The linearization of the above relation by putting x *by* $x + y$ and using (42) gives

(43) $\left[[a, x], y\right] + \left[[a, y], x\right] = 0$

putting y *by* xy in (43) and using (42) , above relation reduces to

$x[[a, x], y] + [a, x][y, x] + x[[a, y], x] = 0$

Using (43) in the above we get

(44) $[a, x][y, x] = 0$

Substituting ya for y in the above relation and using (44) gives

$[a, x]y[a, x] = 0$

By the semiprimeness of S $[a, x] = 0$

$a \in Z(S)$

Now (38) is reduced to $T(x) = ax$

and $T(x) = xa, \; x \in S$

Hence T is a centralizer.

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