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Unified Mittag-Leffler Function and Extended Riemann-Liouville Fractional Derivative Operator

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Abstract

In this paper we study certain properties of extended Riemann-Liouville fractional derivative operator associated with unified Mittag-Leffler function in the form of theorems. Further we have enumerated some results of Generalized Mittag-Leffler function and Riemann-Liouville fractional derivative operator as special cases.

1. Introduction and Preliminaries

Unified Mittag-Leffler function

The Swedish mathematician Gosta Mittag-Leffler [1] introduced the function $E_\alpha(z)$ in 1903 in the form

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\qquad \alpha,z\in C;\ \mathrm{Re}(\alpha)>0 \tag{1}$$

Due to involvement of Mittag Leffler function in problems of Physics, Biology, Engineering, Applied sciences and in the solution of fractional order differential or integral equations, numerous researchers defined and studied various generalizations of Mittag Leffler type functions as $E_{\alpha,\beta}(z)$ by Wimon [2], $E_{\alpha,\beta}^{\gamma}(z)$ by Prabhakar [3], $E_{\alpha,\beta}^{\gamma,q}(z)$ by Shukla and Prajapati [4], $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ by Srivastava and Tomovski [5] and $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ by Salim and Faraz [6].

Recently, Shukla and Prajapati [7] introduced a new generalization of Mittag-Leffler function named Unified Mittag Leffler function in the form

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(cz;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(cz)^{pn+\rho-1}}{\Gamma\left[\alpha(pn+\rho-1)+\beta\right]\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}} \tag{2}$$

where $\alpha,\beta,\gamma,\lambda,\rho,z\in C;\ \mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0;\ \delta,\mu,p,c>0$

and $(\gamma)_{qn}=\dfrac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$, is the generalized Pochhammer symbol.

Due to significant applications in various diverse fields of science and engineering, fractional calculus has gained considerable importance during past four decades [8-14]. Hence in recent years, certain extended fractional derivative operators have been introduced and studied [15, 16]. Recently, Agarwal, Choi and Paris introduced a new extension of Riemann-Liouville fractional derivative operator [19]. In this paper we will study certain properties of this extended Riemann-Liouville fractional derivative operator with unified Mittag-Leffler function.

Before Defining extended Riemann-Liouville fractional derivative operator, we are giving some definitions:

Definition 1.1 The extended Beta function $B_{p}^{\alpha,\beta;\kappa,\mu}(x,y)$ is given as [17]:

$$B_{p}^{\alpha,\beta;\kappa,\mu}(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}{}_{1}F_{1}\left(\alpha;\beta;-\frac{p}{t^{k}(1-t)^{\mu}}\right)dt \tag{3}$$

where $\kappa\geq 0,\ \mu\geq 0,\ R(p)>0,\min\left\{R(\alpha),R(\beta)\right\}>0,R(x)>-R(\kappa\alpha),R(y)>-R(\mu\alpha)$

Definition 1.2 The extended Gauss hypergeometric function is given as [19]:

$$F_{p}^{\alpha,\beta;\kappa,\mu}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_{n}\frac{B_{p}^{\alpha,\beta;\kappa,\mu}(b+n,c-b)}{B(b,c-b)}\frac{z^{n}}{n!} \tag{4}$$

where $|z|<1,\min\left\{R(\alpha),R(\beta),R(\kappa),R(\mu)\right\}>0,\ R(c)>R(b)>0,\ R(p)\geq 0$

Definition 1.3 An extension of the extended Gauss hypergeometric $F_{p}^{\alpha,\beta;\kappa,\mu}$ function is given as [19]:

$$F_{p;\kappa,\mu}(a,b;c;z;m)=\sum_{n=0}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}}\frac{B_{p}^{\alpha,\beta;\kappa,\mu}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^{n}}{n!} \tag{5}$$

where $|z|<1,p\geq 0,R(\kappa)>0,R(\mu)>0,\ m<R(b)<R(c)$

Definition 1.4 An extension of the extended Appell hypergeometric function F_1 is given as [19]:

$$F_{1,p;\kappa,\mu}(a,b,c;d;x,y;m)=\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_{n}(c)_{k}}{(d)_{n+k}}\frac{B_{p}^{\alpha,\beta;\kappa,\mu}(a+n+k,d-a+m)}{B(a+n+k,d-a+m)}\frac{x^{n}}{n!}\frac{y^{k}}{k!} \tag{6}$$

where $|x|<1, |y|<1, p\geq 0, R(\kappa)>0, R(\mu)>0,\ m<R(a)<R(d)$

Definition 1.5 An extension of the Appell hypergeometric function F_2 is given as [19]:

$$F_{2,p;\kappa,\mu}(a,b,c;d,e;x,y;m)$$
$$=\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(c)_k}{(d)_n(e)_k}\frac{B_p^{\alpha,\beta;\kappa,\mu}(b+n,d-b+m)}{B(b+n,d-b+m)}\frac{B_p^{\alpha,\beta;\kappa,\mu}(c+k,e-c+m)}{B(c+k,e-c+m)}\frac{x^n}{n!}\frac{z^k}{k!} \tag{7}$$

where $|x|+|y|<1, p\geq 0, R(\kappa)>0, R(\mu)>0,\ m<R(b)<R(d),\ m<R(c)<R(e)$

Definition 1.6 An extension of the Lauricella hypergeometric function F_D^3 is given as [19]:

$$F_{D,p;\kappa,\mu}^{3}(a,b,c,d;e;x,y,z;m)$$
$$=\sum_{n,k,r=0}^{\infty}\frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(e)_{n+k+r}}\frac{B_p^{\alpha,\beta;\kappa,\mu}(a+n+k+r,e-a+m)}{B(a+n+k+r,e-a+m)}\frac{x^n}{n!}\frac{y^k}{k!}\frac{z^r}{r!} \tag{8}$$

where $|x|<1, |y|<1, |z|<1, p\geq 0, R(\kappa)>0, R(\mu)>0,\ m<R(a)<R(e)$

Definition 1.7 The classical Riemann-Liouville fractional derivative operator of order ν is given as [20]:

$$D_z^{\nu}f(z)=\frac{1}{\Gamma(-\nu)}\int_0^z (z-t)^{-\nu-1}f(t)dt, \qquad R(\nu)\geq 0 \tag{9}$$

and $D_z^{\nu}f(z)=\dfrac{d^m}{dz^m}D_z^{\nu-m}f(z),\quad R(\nu)\geq 0,\ m-1\leq R(\nu)<m$ (10)

Extended Riemann-Liouville fractional derivative operator

The extended Riemann-Liouville fractional derivative operator of order ν is defined as [19]:

$$D_z^{\nu,p;\kappa,\mu}f(z)=\frac{1}{\Gamma(-\nu)}\int_0^z (z-t)^{-\nu-1}f(t)\,{}_1F_1\left(\alpha;\beta;-\frac{pz^{\kappa+\mu}}{t^{\kappa}(z-t)^{\mu}}\right)dt \tag{11}$$

where $R(\nu)<0, R(p)>0, R(\kappa)>0, R(\mu)>0$

For $R(\nu)\geq 0$

$$D_z^{\nu,p;\kappa,\mu}f(z)=\frac{d^m}{dz^m}D_z^{\nu-m,p;\kappa,\mu}f(z) \tag{12}$$

where $m-1\leq R(\nu)<m\left(m\in N\right), R(p)>0, R(\kappa)>0, R(\mu)>0$

2. MAIN RESULTS

Lemma 1. For $R(\nu)\geq 0, m-1\leq R(\nu)<m\left(m\in N\right)$ and $R(\nu)<R(\lambda)$, we have [19]:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\lambda}\right\}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\lambda+1,m-\nu)}{B(\lambda+1,m-\nu)}z^{\lambda-\nu} \tag{13}$$

Theorem 1. For $R(\nu)\geq 0, m-1\leq R(\nu)<m\left(m\in N\right)$ and $R(\nu)<R\left(\beta'-1\right)$, we have:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}\frac{B_p^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta',m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta',m-\nu\right]} \tag{14}$$

Proof. Using definition of unified Mittag-Leffler function, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}\right\}$$

After interchanging the role of summation and extended Riemann-Liouville fractional derivative operator, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(c)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}D_z^{\nu,p;\kappa,\mu}\left\{z^{\alpha'\left(p'n+\rho-1\right)+\beta'-1}\right\}$$

using Lemma 1 and after some simplifications, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}=z^{\beta'-1-\nu}$$

$$\times\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(c)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}\frac{B_p^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta',m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta',m-\nu\right]}z^{\alpha'\left(p'n+\rho-1\right)}$$

which is required result.

Remark 1.1 If we take $p=0$ in equation (14), we obtain this result for unified Mittag-Leffler function and classical Riemann-Liouville fractional derivative operator.

Remark 1.2 If we take $p'=\rho=s=c=1, r=0, \delta=q$ in equation (14), we obtain this result for generalized Mittag-Leffler function.

Theorem 2. For $R(\nu)\geq 0, m-1\leq R(\nu)<m\left(m\in N\right)$ and $R(\nu)<R(\beta'-1)$, we have:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}=$$

$$\times z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[\left(\lambda\right)_{\mu'n}\right]^{r}\left(\rho\right)_{p'n}} \tag{15}$$

$$\times F_{p;\kappa,\mu}\left[\alpha,\alpha'\left(p'n+\rho-1\right)+\beta';\alpha'\left(p'n+\rho-1\right)+\beta'-\nu;z;m\right]$$

Proof. Using definition of unified Mittag-Leffler function, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}\right\}$$

using binomial series expansion, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!}z^l\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}\right\}$$

After interchanging the role of summations and extended Riemann-Liouville fractional derivative operator, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} c^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}} \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!} D_z^{\nu,p;\kappa,\mu}\left\{z^{\alpha'(p'n+\rho-1)+\beta'+l-1}\right\}$$

using Lemma 1 and after some simplifications, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha} E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} c^{p'n+\rho-1} z^{\alpha'(p'n+\rho-1)+\beta'-\nu-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}} \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!} \frac{\left[\alpha'(p'n+\rho-1)+\beta'\right]_l}{\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]_l} \tag{16}$$

$$\times \frac{B_p^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l, m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l, m-\nu\right]} z^l$$

using extension of the extended Gauss hypergeometric function given by equation (5), we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha} E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$= z^{\beta'-\nu-1} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^{s} \left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}$$

$$\times F_{p;\kappa,\mu}\left[\alpha, \alpha'(p'n+\rho-1)+\beta'; \alpha'(p'n+\rho-1)+\beta'-\nu; z; m\right]$$

which is required result.

Remark 2.1 If we take $p = 0$ in equation (14), we obtain this result for unified Mittag-Leffler function and classical Riemann-Liouville fractional derivative operator.

Remark 2.2 If we take $p' = \rho = s = c = 1, r = 0, \delta = q$ in equation (14), we obtain this result for generalized Mittag-Leffler function.

Theorem 3. For $R(\nu) \geq 0, m-1 \leq R(\nu) < m\left(m \in N\right)$ and $R(\nu) < R(\beta'-1)$, we have:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}$$

$$\times F_{1,p;\kappa,\mu}\left[\alpha'(p'n+\rho-1)+\beta',\alpha,\beta;\alpha'(p'n+\rho-1)+\beta'-\nu;az,bz;m\right] \tag{17}$$

Proof. Using definition of unified Mittag-Leffler function, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\right\}$$

using binomial series expansion, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\sum_{l_1,l_2=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_2}}{l_1!\,l_2!}(az)^{l_1}(bz)^{l_2}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\right\}$$

After interchanging the role of summations and extended Riemann-Liouville fractional derivative operator, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s c^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\sum_{l_1,l_2=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_2}}{l_1!\,l_2!}(a)^{l_1}(b)^{l_2}D_z^{\nu,p;\kappa,\mu}\left\{z^{\alpha'(p'n+\rho-1)+\beta'+l_1+l_2-1}\right\}$$

using Lemma 1 and after some simplifications, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}c^{p'n+\rho-1}z^{\alpha'(p'n+\rho-1)+\beta'-\nu-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu' n}\right]^{r}(\rho)_{p'n}}\sum_{l_1,l_2=0}^{\infty}\frac{\left[\alpha'(p'n+\rho-1)+\beta'\right]_{l_1+l_2}(\alpha)_{l_1}(\beta)_{l_2}}{\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]_{l_1+l_2}}$$

$$\times\frac{B_{p}^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1+l_2,m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1+l_2,m-\nu\right]}\frac{(az)^{l_1}}{l_1!}\frac{(bz)^{l_2}}{l_2!}$$

using extension of the extended Appell hypergeometric function given by equation (6), we get:

$$D_{z}^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-az\right)^{-\alpha}\left(1-bz\right)^{-\beta}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[(\lambda)_{\mu' n}\right]^{r}(\rho)_{p'n}}$$

$$\times F_{1,p;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta',\alpha,\beta;\alpha'\left(p'n+\rho-1\right)+\beta'-\nu;az,bz;m\right]$$

which is required result.

Remark 3.1 If we take $p=0$ in equation (14), we obtain this result for unified Mittag-Leffler function and classical Riemann-Liouville fractional derivative operator.

Remark 3.2 If we take $p'=\rho=s=c=1, r=0, \delta=q$ in equation (14), we obtain this result for generalized Mittag-Leffler function.

Theorem 4. For $R(\nu)\geq 0, m-1\leq R(\nu)<m\left(m\in N\right)$ and $R(\nu)<R(\beta'-1)$, we have:

$$D_{z}^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-az\right)^{-\alpha}\left(1-bz\right)^{-\beta}\left(1-dz\right)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[(\lambda)_{\mu' n}\right]^{r}(\rho)_{p'n}}$$

$$\times F_{D,p;\kappa,\mu}^{3}\left[\alpha'\left(p'n+\rho-1\right)+\beta',\alpha,\beta,\tau;\alpha'\left(p'n+\rho-1\right)+\beta'-\nu;az,bz,dz;m\right] \quad (18)$$

Proof. Using definition of unified Mittag-Leffler function, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-dz)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-dz)^{-\tau}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\right\}$$

using binomial series expansion, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-dz)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\sum_{l_1,l_2,l_3=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_2}(\tau)_{l_3}}{l_1!\,l_2!l_3}(az)^{l_1}(bz)^{l_2}(dz)^{l_3}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\right\}$$

After interchanging the role of summations and extended Riemann-Liouville fractional derivative operator, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-dz)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s c^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\sum_{l_1,l_2,l_3=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_2}(\tau)_{l_3}}{l_1!\,l_2!l_3}(a)^{l_1}(b)^{l_2}(d)^{l_3}$$

$$\times D_z^{\nu,p;\kappa,\mu}\left\{z^{\alpha'(p'n+\rho-1)+\beta'+l_1+l_2+l_3-1}\right\}$$

using Lemma 1 and after some simplifications, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-az\right)^{-\alpha}\left(1-bz\right)^{-\beta}\left(1-dz\right)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}c^{p'n+\rho-1}z^{\alpha'(p'n+\rho-1)+\beta'-\nu-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^{r}(\rho)_{p'n}}$$

$$\times\sum_{l_1,l_2,l_3=0}^{\infty}\frac{\left[\alpha'(p'n+\rho-1)+\beta'\right]_{l_1+l_2+l_3}(\alpha)_{l_1}\left(\beta\right)_{l_2}\left(\tau\right)_{l_3}}{\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]_{l_1+l_2+l_3}}$$

$$\times\frac{B_p^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1+l_2+l_3,m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1+l_2+l_3,m-\nu\right]}\frac{\left(az\right)^{l_1}}{l_1!}\frac{\left(bz\right)^{l_2}}{l_2!}\frac{\left(dz\right)^{l_3}}{l_3!}$$

using extension of the extended Lauricella hypergeometric function given by equation (8), we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-az\right)^{-\alpha}\left(1-bz\right)^{-\beta}\left(1-dz\right)^{-\tau}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[\left(\lambda\right)_{\mu'n}\right]^{r}\left(\rho\right)_{p'n}}$$

$$\times F_{D,p;\kappa,\mu}^{3}\left[\alpha'\left(p'n+\rho-1\right)+\beta',\alpha,\beta,\tau;\alpha'\left(p'n+\rho-1\right)+\beta'-\nu;az,bz,dz;m\right]$$

which is required result.

Remark 4.1 If we take $p=0$ in equation (14), we obtain this result for unified Mittag-Leffler function and classical Riemann-Liouville fractional derivative operator.

Remark 4.2 If we take $p'=\rho=s=c=1,r=0,\delta=q$ in equation (14), we obtain this result for generalized Mittag-Leffler function.

Theorem 5. For $R(\nu)\geq 0, m-1\leq R(\nu)<m\left(m\in N\right)$ and $R(\nu)<R(\beta'-1)$, we have:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\beta;\tau;\frac{x}{1-z};m\right)E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}$$

$$\times F_{2,p;\kappa,\mu}\left[\alpha,\beta,\alpha'(p'n+\rho-1)+\beta';\tau,\alpha'(p'n+\rho-1)+\beta'-\nu;x,z;m\right] \quad (19)$$

Proof. Using definition of extension of the extended Gauss hypergeometric function, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\beta;\tau;\frac{x}{1-z};m\right)E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$
$$=D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\sum_{l_1=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_1}}{(\tau)_{l_1}l_1!}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+l_1,\tau-\beta+m)}{B(\beta+l_1,\tau-\beta+m)}\left(\frac{x}{1-z}\right)^{l_1}\right\}$$

After interchanging the role of summation and extended Riemann-Liouville fractional derivative operator, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\beta;\tau;\frac{x}{1-z};m\right)E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=\sum_{l_1=0}^{\infty}\frac{(\alpha)_{l_1}(\beta)_{l_1}x^{l_1}}{(\tau)_{l_1}l_1!}\frac{B_p^{\alpha,\beta;\kappa,\mu}(\beta+l_1,\tau-\beta+m)}{B(\beta+l_1,\tau-\beta+m)}D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha-l_1}E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

using result of Theorem 2 given by equation (16) and after some simplifications, we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}(1-z)^{-\alpha}F_{p;\kappa,\mu}\left(\alpha,\beta;\tau;\frac{x}{1-z};m\right)E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$=z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'(p'n+\rho-1)+\beta'-\nu\right]\left[(\lambda)_{\mu'n}\right]^r(\rho)_{p'n}}\sum_{l_1,l_2=0}^{\infty}\frac{(\alpha)_{l_1+l_2}(\beta)_{l_1}\left[\alpha'(pn+\rho-1)+\beta'\right]_{l_2}}{(\tau)_{l_1}\left[\alpha'(pn+\rho-1)+\beta'-\nu\right]_{l_2}}$$

$$\times \frac{B_p^{\alpha,\beta;\kappa,\mu}\left(\beta+l_1,\tau-\beta+m\right)}{B\left(\beta+l_1,\tau-\beta+m\right)} \frac{B_p^{\alpha,\beta;\kappa,\mu}\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1,m-\nu\right]}{B\left[\alpha'\left(p'n+\rho-1\right)+\beta'+l_1,m-\nu\right]} \frac{x^{l_1}}{l_1!} \frac{z^{l_2}}{l_2!}$$

using extension of the extended Appell hypergeometric function given by equation (7), we get:

$$D_z^{\nu,p;\kappa,\mu}\left\{z^{\beta'-1}\left(1-z\right)^{-\alpha} F_{p;\kappa,\mu}\left(\alpha,\beta;\tau;\frac{x}{1-z};m\right) E_{\alpha',\beta',\lambda,\mu',\rho,p'}^{\gamma,\delta}\left(cz^{\alpha'};s,r\right)\right\}$$

$$= z^{\beta'-\nu-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(cz^{\alpha'}\right)^{p'n+\rho-1}}{\Gamma\left[\alpha'\left(p'n+\rho-1\right)+\beta'-\nu\right]\left[\left(\lambda\right)_{\mu'n}\right]^r\left(\rho\right)_{p'n}}$$

$$\times F_{2,p;\kappa,\mu}\left[\alpha,\beta,\alpha'\left(p'n+\rho-1\right)+\beta';\tau,\alpha'\left(p'n+\rho-1\right)+\beta'-\nu;x,z;m\right]$$

which is required result.

Remark 5.1 If we take $p=0$ in equation (14), we obtain this result for unified Mittag-Leffler function and classical Riemann-Liouville fractional derivative operator.

Remark 5.2 If we take $p'=\rho=s=c=1, r=0, \delta=q$ in equation (14), we obtain this result for generalized Mittag-Leffler function.

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