Proof for Goldbach’s Strong Conjecture

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Abstract
In this paper, we will give a simple solution/proof for the Goldbach’s strong conjecture (called the “strong” or “binary”).

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1. INTRODUCTION
Goldbach's original conjecture (sometimes called the "ternary" Goldbach conjecture), written in a June 7, 1742 letter to Euler, states "at least it seems that every number that is greater than 2 is the sum of three primes" (Goldbach 1742; Dickson 2005). Note that here Goldbach considered the number 1 to be a prime, a convention that is no longer followed. As re-expressed by Euler, an equivalent form of this conjecture (called the "strong" or "binary" Goldbach conjecture) asserts that all positive even integers ≥ 4 can be expressed as the sum of two primes. Two primes (p, q) such that \( p + q = 2n \) for \( n \) a positive integer are sometimes called a Goldbach partition [1].

In this paper, we will give a simple solution to the Goldbach’s strong conjecture. We will first make a general formula for primes (>2), and then evaluate all possibilities.

2. METHODS
Lemma: Every prime number can be written in the form \( 4k \pm 1 \); where \( k \in \mathbb{I}^+ \)

Proof:
A number can be expressed as \( (4k) \) or \( (4k + 1) \) or \( (4k + 2) \) or \( (4k + 3) \)(or \( 4k - 1 \)).
Since, all prime numbers (>2) are odd, therefore, we can rule out \((4k)\) and \((4k + 2)\) as a possible expression for primes.

Thus, the only possible representations for primes are \((4k + 1)\) or \((4k - 1)\); \(k \in I^+\).

Let’s say we have 4 consecutive numbers \((a), (a + 1), (a + 2), (a + 3)\).

Out of these four, there will be 2 alternate even numbers, and out of them, at least one will be divisible by 4 (as out of every 4 consecutive numbers, one of them will definitely be divisible by 4, and it has to be even).

Let the even numbers be \((a + 1)\) and \((a + 3)\).

**CASE I:** \((a + 1)\) is divisible by 4

If \((a + 2)\) is a prime number, it can be written as \((4p + 1)\); where \((a + 1) = 4p; p \in I^+\).

AND

If \((a)\) is a prime number, it can be written as \((4p - 1)\); where \((a + 1) = 4p; p \in I^+\).

**CASE II:** \((a + 3)\) is divisible by 4

If \((a + 2)\) is a prime number, it can be written as \((4q - 1)\); where \((a + 3) = 4q; q \in I^+\).

AND

If \((a)\) is a prime number, it can be written as \((4(q - 1) + 1)\); where \((a + 3) = 4q and (a - 1) = 4(q - 1); q \in I^+\).

[since \((a + 3)\) is a multiple of 4 ⇒ \{\((a + 3) - 4 = (a - 1)\}\} is also a multiple of 4]

Similarly, the same reasoning can be applied when taking \((a)\) and \((a + 2)\) as even.

Thus, prime numbers (>2) are of the form \((4k \pm 1)\); where \(k \in I^+\).

**Remark:** Thus, every prime number, \(p (>2)\), can be written as:

\[ p = (4k + 1) or (4k - 1) = (4k \pm 1); where k \in I^+ \]

Examples: \(3 = (4 \times 1) - 1,\)
\(5 = (4 \times 1) + 1,\)
\(7 = (4 \times 2) - 1,\)
\(17389 = (4 \times 4347) + 1,\) and so on.
**Proof for Goldbach’s Strong Conjecture**

**Conjecture:** Every even integer greater than 2 can be expressed as the sum of two primes. [2]

**Proof:**

Let there be 2 prime numbers (>2), namely, $p_1$ and $p_2$.

\[ p_1 = 4n \pm 1; n \in I^+ \] \hspace{1cm} (1)

\[ p_2 = 4m \pm 1; m \in I^+ \] \hspace{1cm} (2)

\[ \Rightarrow p_1 + p_2 = (4n \pm 1) + (4m \pm 1) \]

Thus, we have only 4 possibilities:

I. \((4n + 1) + (4m + 1)\)

II. \((4n + 1) + (4m - 1)\)

III. \((4n - 1) + (4m + 1)\)

IV. \((4n - 1) + (4m - 1)\)

Here, possibilities I and IV (same sign of 1s), and II and III (opposite signs of 1s) are similar.

**CASE I:** Considering possibilities I and IV

Therefore,

\[ p_1 + p_2 = (4n + 1) + (4m + 1) \text{ or } p_1 + p_2 = (4n - 1) + (4m - 1) \]

\[ \Rightarrow p_1 + p_2 = 4(m + n) \pm 2 \]

\[ \Rightarrow p_1 + p_2 = 2[2(m + n) \pm 1] \]

\[ \Rightarrow p_1 + p_2 = 2f; \text{ where } f = [2(m + n) \pm 1] \]

Thus, \(f\) is a positive odd integer (\(\because\) it is of the form \(2q \pm 1\) except 1 (\(\because\) minimum value of \(m\) and \(n\) = 1[from (1) and (2)] \(\Rightarrow\) minimum value of \([2(m + n) \pm 1] = 3\))

\[ \Rightarrow f \in \{3, 5, 7, \ldots\} \]

**CASE II:** Considering possibilities II and III

Therefore,

\[ p_1 + p_2 = (4n + 1) + (4m - 1) \text{ or } p_1 + p_2 = (4n - 1) + (4m + 1) \]

\[ \Rightarrow p_1 + p_2 = 4(m + n) \]

\[ \Rightarrow p_1 + p_2 = 2[2(m + n)] \]

\[ \Rightarrow p_1 + p_2 = 2f'; \text{ where } f' = [2(m + n)] \]
Thus, \( f' \) is a positive even integer (\( \because \) it is of the form \( 2q \)) except 2 (\( \because \) minimum value of \( m \) and \( n = 1 \) \( \Rightarrow \) minimum value of \( [2(m + n)] = 4 \))

\[ \therefore f' \in \{4, 6, 8, \ldots \} \]

Hence, on combining cases I and II, we obtain

\[ p_1 + p_2 = 2F ; \text{where } F \in f \cup f' , \text{i.e. } F \in I^+ - \{1, 2\} \]

\[ \therefore F \in \{3, 4, 5, 6, \ldots \} \]

Thus, \( F \) is a positive integer except 1 and 2, and minimum value of \( 2F = 6 \).

This means that every even integer (\( 2F \in \{6, 8, 10, \ldots \} \)) will have at least one pair of primes (\( p_1 \) and \( p_2 \)) such that \( p_1 + p_2 = 2F \)

Thus, every even integer (\( >4 \)) can be expressed as the sum of two primes.

The only case we are left with is – the even integer 4. We know that 4 can be expressed as the sum of two 2s, i.e. \( 4=2+2 \), thus, covering all the even integers (since 2 is a prime number, therefore, 4 is the sum of two primes).

3. CONCLUSION

Therefore, every even integer (\( >4 \)) can be expressed as the sum of two primes (of the form \( 4k \pm 1 \); where \( k \in I^+ \)), and the even integer 4 can be expressed as the sum of two 2s. Thus, every even integer greater than 2 can be expressed as the sum of two primes, and hence, proving the Goldbach’s Strong Conjecture.

4. REFERENCES
