International Journal of Mathematics Research.
ISSN 0976-5840 Volume 8, Number 1 (2016), pp. 61-69

http://www.irphouse.com

The Multiplicative Zagreb Indices of Products of Graphs

R. Muruganandam[1], R. S. Manikandan[2] and Aruvi M.[3]

[1]*Department of Mathematics, Government Arts College, Trichy, Tamil Nadu, India.*

[2]*Department of Mathematics, Bharathidasan Constituent College, Lalgudi, Trichy, Tamil Nadu, India.*

[3]*Department of Mathematics, Anna University – BIT Campus, Trichy, Tamil Nadu, India.*

Abstract

In this paper, we determine the exact formulae for the multiplicative Zagreb indices of strong and tensor products of two connected graphs. Also we apply our results to compute the multiplicative Zagreb indices of open and closed fence graphs.

Keywords: The multiplicative Zagreb indices, strong product and tensor product.

1. Introduction

Throughout this paper, we consider simple graphs which are finite, indirected graphs without loops and multiple edges. Suppose G is a graph with vertex set $V(G)$ and an edge set $E(G)$. For a graph G, the degree of a vertex v is the number of edges incident to v and is denoted by $d_G(v)$. A topological index $Top(G)$ of a graph G is a number with the property that for every graph H is isomorphic to G , $Top(H) = Top(G)$.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Frianjestic [2]. They are defined as $M_1(G) = \sum_{u\in V(G)}(d_G(u))^2 = \sum_{uv\in E(G)}[d_G(u) + d_G(v)]$ and $M_2(G) = \sum_{uv\in E(G)}[d_G(u)d_G(v)]$.

The Zagreb indices are found to have applications in QSPR and QSAR studies as well see [3]. Recently, Todeschini et al. [4, 5] have proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$\prod_1 = \prod_1(G) = \prod_{u\in V(G)}[d_G(u)]^2 = \prod_{uv\in V(G)}[d_G(u) + d_G(v)]$

and $\prod_2 = \prod_2(G) = \sum_{uv\in E(G)}[d_G(u)d_G(v)]$

Mathematical properties and applications of multiplicative Zagreb indices are reported in [4-9].Mathematical properties and applications of multiplicative sum Zagreb indices are reported in [10]. In [11], K.Pattabiraman computed Zagreb indices and coindices of product graphs.

The Strong product of graphs $G_1$and G_2,denoted by $G_1 \otimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2) = \{(u,v): u \in V(G_1), v \in V(G_2)\}$ and $(u,x)(v,y)$ is an edge whenever (i) $u = v$ and $xy \in E(G_2)$ or $(ii) uv \in E(G_1)$ and $x = y$ or (iii) $uv \in E(G_1)$ and $xy \in E(G_2)$. For two simple graphs G_1 and G_2 their tensor product, denoted by $G_1 \times G_2$, has the vertex set $V(G_1) \times V(G_2)$, in which (u,v) and (x,y) are adjacent whenever ux is an egde in G_1 and vy is an edge in G_2. Note that if G_1 and G_2 are connected graphs, then $G_1\, X\, G_2$ is connected only if atleast one of the graph is nonbipartite.

In this paper, we compute the multiplicative Zagreb indices of strong and tensor products of two connected graphs. Moreover, computations are done for some well known graphs.

2. The Multiplicative Zagreb Indices of $G_1 \otimes G_2$

We begin this section with standard inequality as follows:

Lemma – 2.1 (Arithmetic Geometric Inequality)

Let $x_1, x_2, \dots, x_n$ be non-negative numbers. Then $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots\ldots x_n}$ holds with equality if and only if all x_k's are equal.

In this section, we compute the multiplicative Zagreb indices of the strong product of graphs. The following lemma follows from the definition of the strong product of the two graphs.

Lemma – 2.2

Let G_1 and G_2be two graphs.

$(i). |V(G_1 \otimes G_2)| = |V(G_1)||V(G_2)|$

$(ii). |E(G_1 \otimes G_2)| = |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)| + 2|E(G_1)||E(G_2)|$

$(iii).\ \ d_{G_1 \otimes G_2}(u_i, u_j) = d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j).$

Theorem – 2.3

Suppose $G_1$and G_2 are graphs with $|V(G_1)| = n_1, |V(G_2)| = n_2, |E(G_1)| = m_1$ and $|E(G_2)| = m_2$.

Then

$$\prod_1(G_1 \otimes G_2) \leq \left[\frac{(n_1+4m_2)M_1(G_1)+(n_2+4m_1)M_1(G_2)+M_1(G_1)M_1(G_2)+8m_1m_2}{n_1n_2}\right]^{n_1n_2}$$

where $M_1(G_1)$ is the first Zagreb index of G.

Proof:

By the definition of the multiplicative first Zagreb index,

$$\prod_1(G_1 \otimes G_2) = \prod_{(u_i,v_j)\in V(G_1\otimes G_2)}\left[d_{G_1\otimes G_2}(u_i, v_j)\right]^2$$

$$= \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}\left[d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j)\right]^2 \quad \text{by lemma 2.2}$$

$$\leq \left[\frac{\sum_{u_i\in V(G_1)}\sum_{v_j\in V(G_2)}\left[\begin{array}{c} d^2_{G_1}(u_i)+d^2_{G_2}(v_j)+d^2_{G_1}(u_i)d^2_{G_2}(v_j)+2d_{G_1}(u_i)d_{G_2}(v_j) \\ +2d^2_{G_1}(u_i)d_{G_2}(v_j)+2d_{G_1}(u_i)d^2_{G_2}(v_j)\end{array}\right]}{n_1n_2}\right]^{n_1n_2} \quad \text{by lemma 2.1}$$

$$\prod_1(G_1 \otimes G_2) \leq \left[\frac{(n_1+4m_2)M_1(G_1)+(n_2+4m_1)M_1(G_2)+M_1(G_1)M_1(G_2)+8m_1m_2}{n_1n_2}\right]^{n_1n_2}$$

by the definition of the first Zagreb index of G.

In $G_1 \otimes G_2$, define

$E_1 = \{(u, v)(x, y) \in E(G_1 \otimes G_2) | ux \in E(G_1)\ and\ v = y\}$,

$E_2 = \{(u, v)(x, y) \in E(G_1 \otimes G_2) | ux \in E(G_1)\ and\ vy \in E(G_2)\}$,

$E_3 = \{(u, v)(x, y) \in E(G_1 \otimes G_2) | u = x\ and\ vy \in E(G_2)\}$.

Clearly, $E_1 \cup E_2 \cup E_3 = E(G_1 \otimes G_2)$.

Also $|E_1| = |E(G_1)||V(G_2)|, |E_2| = 2|E(G_1)||E(G_2)|\ and\ |E_3| = |E(G_2)||V(G_1)|$.

Theorem 2.4

Let $G_1$and G_2 be two graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then

$$\prod_2(G_1 \otimes G_2) \leq \frac{1}{(m_1n_2)^{m_1n_2}}[n_2M_2(G_1) + 2m_2M_1(G_1) + m_1M_1(G_2) + 4m_2M_2(G_1) + M_1(G_1)M_1(G_2) + M_2(G_1)M_1(G_2)]^{m_1n_2}\ X$$

$$\frac{2}{(m_1m_2)^{m_1m_2}}[m_2M_2(G_1) + m_1M_2(G_2) + M_2(G_1)M_2(G_2) + M_2(G_1)M_1(G_2) + M_1(G_1)M_2(G_2) + 8m_1m_2]^{m_1m_2}\ X$$

$$\frac{1}{(m_2n_1)^{m_2n_1}}[m_2M_1(G_1) + 2m_1M_1(G_2) + n_1M_2(G_2) + 4m_1M_2(G_2) + M_1(G_1)M_1(G_2) + M_1(G_1)M_2(G_2)]^{m_2n_1}$$

where $M_1(G)$ and $M_2(G)$ are the first and second Zagreb indices respectively.

Proof:

From the above partition of the edge set in $G_1 \otimes G_2$, we have

$$\begin{aligned}\prod_2 G_1 \otimes G_2 &= \prod_{(u_i,v_j)(u_p,v_q)\in E(G_1\otimes G_2)} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_p,v_q)\\ &= \prod\nolimits_{(u_i,v_j)(u_p,v_j)\in E_1} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_p,v_j)\ X\\ &\quad \prod\nolimits_{(u_i,v_j)(u_p,v_q)\in E_2} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_p,v_q)\ X\\ &\quad \prod\nolimits_{(u_i,v_j)(u_i,v_q)\in E_3} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_i,v_q)\\ &= A\ X\ B\ X\ C \qquad (2.1)\end{aligned}$$

where A, B and C are the products of the terms of the above expressions, in order. Now we calculate

$$A = \prod\nolimits_{(u_i,v_j)(u_p,v_j)\in E_1} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_p,v_j)$$

$$= \prod\nolimits_{u_iu_p\in E(G_1)}\prod\nolimits_{v_j\in V(G_2)} \begin{matrix}[d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j)]\\ [d_{G_1}(u_p) + d_{G_2}(v_j) + d_{G_1}(u_p)d_{G_2}(v_j)]\end{matrix}$$

$$\leq \left[\frac{\prod_{u_iu_p\in E(G_1)}\begin{bmatrix}n_2d_{G_1}(u_i)d_{G_1}(u_p)+2m_2[d_{G_1}(u_i)+d_{G_1}(u_p)]+M_1(G_2)+4m_2d_{G_1}(u_i)d_{G_1}(u_p)+\\ M_1(G_2)[d_{G_1}(u_i)+d_{G_1}(u_p)]+M_1(G_2)d_{G_1}(u_i)d_{G_1}(u_p)\end{bmatrix}}{n_2}\right]^{n_2}$$

by the definition of the first Zagreb index.

$$A \leq \frac{1}{(m_1n_2)^{m_1n_2}}[n_2M_2(G_1) + 2m_2M_1(G_1) + m_1M_1(G_2) + 4m_2M_2(G_1) + M_1(G_1)M_1(G_2) + M_2(G_1)M_1(G_2)]^{m_1n_2} \qquad (2.2)$$

by the definition of the second Zagreb index.
Next we calculate

$$B = \prod_{(u_i,v_j)(u_p,v_q)\in E_2} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_p,v_q)$$

$$= 2\prod_{u_iu_p\in E(G_1)}\prod_{v_jv_q\in E(G_2)} \begin{matrix}[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_i)d_{G_2}(v_j)]\\ [d_{G_1}(u_p)+d_{G_2}(v_q)+d_{G_1}(u_p)d_{G_2}(v_q)]\end{matrix}$$

$$\leq 2\left[\frac{\sum_{u_iu_p\in E(G_1)}\sum_{v_jv_q\in E(G_2)}\begin{bmatrix}d_{G_1}(u_i)d_{G_1}(u_p)+d_{G_1}(u_i)d_{G_2}(v_q)+d_{G_1}(u_i)d_{G_1}(u_p)d_{G_2}(v_q)+\\ d_{G_1}(u_p)d_{G_2}(v_j)+d_{G_2}(v_j)d_{G_2}(v_q)+d_{G_1}(u_p)d_{G_2}(v_j)d_{G_2}(v_q)\\ +d_{G_1}(u_i)d_{G_1}(u_p)d_{G_2}(v_j)+d_{G_1}(u_i)d_{G_2}(v_j)d_{G_2}(v_q)\end{bmatrix}}{m_1m_2}\right]^{m_1m_2}$$

$$B \leq \frac{2}{(m_1m_2)^{m_1m_2}}[m_2M_2(G_1)+m_1M_2(G_2)+M_2(G_1)M_2(G_2)+M_2(G_1)M_1(G_2) + M_1(G_1)M_2(G_2)+8m_1m_2]^{m_1m_2}$$

by the definitions of the first and second Zagreb indices of G.
Finally we compute,

$$C = \prod_{(u_i,v_j)(u_i,v_q)\in E_3} d_{G_1\otimes G_2}(u_i,v_j)d_{G_1\otimes G_2}(u_i,v_q)$$

$$= \prod_{u_i\in V(G_1)}\prod_{v_jv_q\in E(G_2)} \begin{matrix}[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_i)d_{G_2}(v_j)]\\ [d_{G_1}(u_i)+d_{G_2}(v_q)+d_{G_1}(u_i)d_{G_2}(v_q)]\end{matrix}$$

$$\leq \left[\frac{\sum_{u_i\in V(G_1)}\sum_{v_jv_q\in E(G_2)}\begin{bmatrix}d^2_{G_1}(u_i)+d_{G_1}(u_i)[d_{G_2}(v_j)+d_{G_2}(v_q)]+d^2_{G_1}(u_i)[d_{G_2}(v_j)+d_{G_2}(v_q)]+\\ d_{G_2}(v_j)d_{G_2}(v_q)+d^2_{G_1}(u_i)d_{G_2}(v_j)d_{G_2}(v_q)+2d_{G_1}(u_i)d_{G_2}(v_j)d_{G_2}(v_q)\end{bmatrix}}{m_2n_1}\right]^{m_2n_1}$$

$$C \leq \frac{1}{(m_2n_1)^{m_2n_1}}[m_2M_1(G_1)+2m_1M_1(G_2)+n_1M_2(G_2)+4m_1M_2(G_2)+ M_1(G_1)M_1(G_2)+M_1(G_1)M_2(G_2)]^{m_2n_1} \tag{2.4}$$

by the definitions of the first and second Zagreb indices of G.
Using (2.2), (2.3) and (2.4) in (2.1), we get

$$\prod_2 (G_1 \otimes G_2) \leq \frac{1}{(m_1 n_2)^{m_1 n_2}} [n_2 M_2(G_1) + 2m_2 M_1(G_1) + m_1 M_1(G_2) + 4m_2 M_2(G_1) + M_1(G_1)M_1(G_2) + M_2(G_1)M_1(G_2)]^{m_1 n_2} \ X$$
$$\frac{2}{(m_1 m_2)^{m_1 m_2}} [m_2 M_2(G_1) + m_1 M_2(G_2) + M_2(G_1)M_2(G_2) + M_2(G_1)M_1(G_2) + M_1(G_1)M_2(G_2) + 8m_1 m_2]^{m_1 m_2} \ X$$
$$\frac{1}{(m_2 n_1)^{m_2 n_1}} [m_2 M_1(G_1) + 2m_1 M_1(G_2) + n_1 M_2(G_2) + 4m_1 M_2(G_2) + M_1(G_1)M_1(G_2) + M_1(G_1)M_2(G_2)]^{m_2 n_1}$$

By direct calculations, we obtain the following expressions as lemma:
Lemma 2.4
Let P_n and C_n denote the path and cycle on n vertices, respectively.
$(1). For\ n \geq 2, M_1(P_n) = 4n - 6 \ \ and \ \ M_1(P_n) = 0$
$(2). For\ \ n \geq 3, \qquad M_1(C_n) = 4n, M_2(C_n) = 4n\ and\ M_2(P_n) = 4(n-2)$
$(3). For\ n \geq 3, M_1(K_n) = n(n-1)^2\ and\ M_2(K_n) = \dfrac{n(n-1)^3}{2}$

By using Theorem – 2.3, and Lemma – 2.4, we obtain the exact multiplicative Zagreb indices of open and closed fences $P_n \otimes K_2$ and $C_n \otimes K_2$, see Fig 2.1.

$$(i). \prod_1 (P_n \otimes K_2) \ \leq \left[\frac{2n^2 + 17n - 24}{n}\right]^n$$

$$(ii). \prod_1 (C_n \otimes K_2) \ \leq \left[\frac{2n^2 + 24n + 2}{n}\right]^{2n}$$

$(iii). \prod_2(P_n \otimes K_2) \leq \frac{1}{(2(n-1))^{2(n-1)}}[50n-90]^{2(n-1)} X \frac{2}{(n-1)^{(n-1)}}[29n-47]^{(n-1)} X \frac{1}{n^n}[25n-34]^n$

$$(iv). \prod_2 (C_n \otimes K_2) \leq \frac{1}{(2n)^{2n}}[50n]^{2n} X \frac{2}{n^n}[30n]^n X \frac{1}{n^n}[25n]^n$$

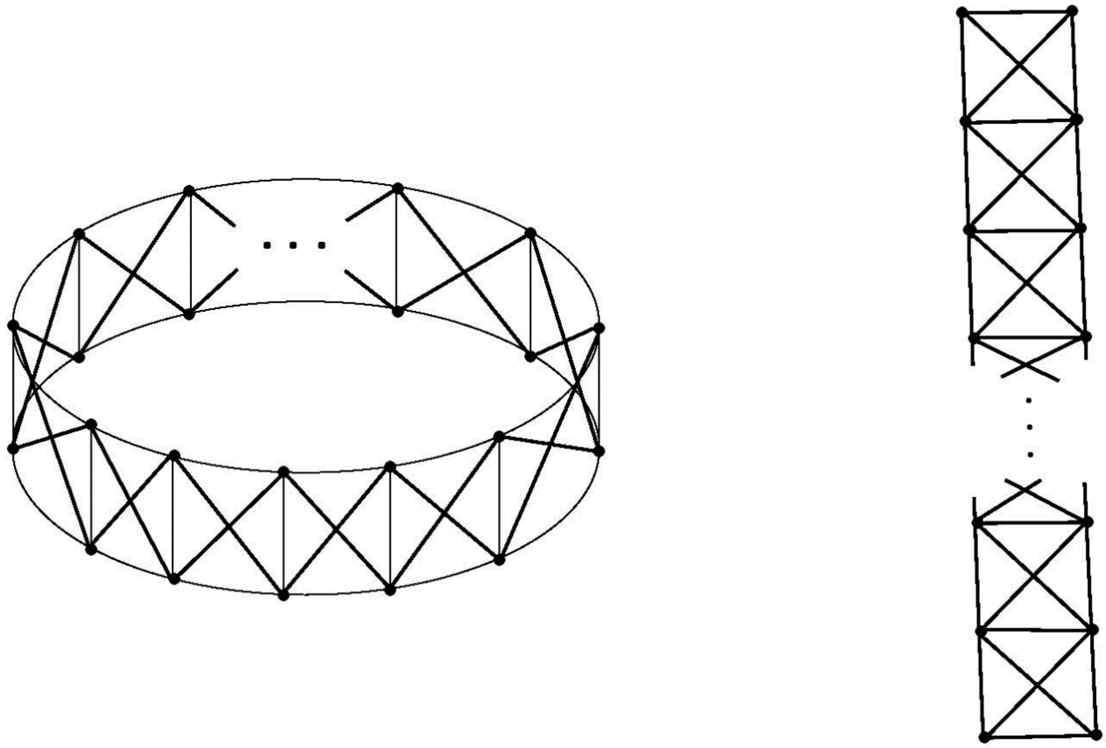

Figure 2.1. Closed and Open fence graphs

3. The Multiplicative Zagreb Indices of G_1XG_2

In this section, we compute the multiplicative Zagreb indices of the strong product of graphs. The following lemma follows from the structure of G_1XG_2.

Lemma – 3.1

Let G_1 and G_2be two graphs. Then

$(i). |V(G_1XG_2)| = |V(G_1)||V(G_2)|$

$(ii). |E(G_1XG_2)| = 2|E(G_1)||E(G_2)|$

(iii). The degree of a vertex (u_i, v_j) of G_1XG_2 is given by $d_{G_1XG_2}(u_i, v_j) = d_{G_1}(u_i)d_{G_2}(v_j)$.

Theorem – 3.2

Let G_1 and G_2be two graphs. Then $\prod_1(G_1XG_2) = \prod_1(G_1)\prod_1(G_2)$, where $\prod_1(G_1)$ and $\prod_1(G_2)$ are the first multiplicative Zagreb index of $G_1$and G_2respectively.

Proof:

By the definition of the first multiplicative Zagreb index,

$$\prod_1(G_1XG_2) = \prod_{(u_i,v_j)\in V(G_1XG_2)} [d_{(G_1XG_2)}(u_i, v_j)]^2$$

$$= \prod_{u_i\in V(G_1)} [d_{G_1}(u_i)]^2 \prod_{v_j\in V(G_2)} [d_{G_2}(v_j)]^2$$

$$= \prod_1(G_1)\prod_1(G_2)$$

by the definition of the first multiplicative Zagreb index of the graphs $G_1$and G_2respectively.

Theorem – 3.3

Let G_1 and G_2be two graphs. Then $\prod_2(G_1XG_2) = 2\prod_2(G_1)\prod_2(G_2)$, where $\prod_2(G_1)$ and $\prod_2(G_2)$ are the second multiplicative Zagreb index of $G_1$and G_2respectively.

Proof:

By the definition of the seond multiplicative Zagreb index,

$$\prod_1(G_1XG_2) = \prod_{(u_i,v_j)(u_p,v_q)\in E(G_1XG_2)} [d_{(G_1XG_2)}(u_i, v_j)][d_{(G_1XG_2)}(u_p, v_q)]$$

$$= 2\prod_{u_iu_p\in E(G_1)}\prod_{v_jv_q\in E(G_2)} d_{G_1}(u_i)\, d_{G_2}(v_j)d_{G_1}(u_p)d_{G_2}(v_q)$$

$$= 2 \prod_{u_i u_p \in E(G_1)} d_{G_1}(u_i) d_{G_1}(u_p) \prod_{v_j v_q \in E(G_2)} d_{G_2}(v_j) d_{G_2}(v_q)$$

$$= 2 \prod_2 (G_1) \prod_2 (G_2)$$

by the definition of the second multiplicative Zagreb index of the graphs $G_1$and G_2respectively.

References

[1] R.Balakrishnan and K.Ranganathan, A Text Book of Graph Theory, Springer-Verlog, New York, 2000.
[2] I.Gutman, N.Trinajstic, Graph Theory and molecular orbits. Total π -election energy of alternate hydrocarbons, chem, Phy. Lett. 17(1972); 535 - 538.
[3] J.Devillers, A.T.Balaban,Eds., Topological indices and related descriptors in QSAR and QSPR, Gordon and Breach, Amstersam, The Netherlands, 1999.
[4] R.Todeschini, D.Ballabio, V.Consonni: Novel molecular descriptors based on functions of new vertex degrees. In: Gutman.I, Furtula.B (eds)Novel molecular structure descriptors - Theory and Applications I, pp.73 - 100. Univ. kragujeva, Kragujevac (2010).
[5] R.Todeschini, V.Consonni, New local vertex invariants and molecular descriptors based on functions of the vertex degrees. MATCH Commun. Math. Comput. Chem. 64, 359 - 372(2010).
[6] M.Eliasi, A.Iranmanesh, I.Gutman, Multiplicative versions of first Zagreb index. MATCH Commun. Math. Comput. Chem. 68, 217 - 230(2012).
[7] I.Gutman, Multiplicative Zagreb indices of trees. Bull. Soc. Math. Banja Luka 18, 17 - 23(2011).
[8] J.Liu, Q.Zhang: Sharp upper bounds on multiplicative Zagreb indices. MATCH Commun. Math. Com-put. Chem. 68, 231 - 240(2012).
[9] Xu.K,Hua.H, A uni_ed approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs. MATCH Commun. Math. Comput. Chem. 68, 241 - 256(2012).
[10] Xu,K.Das,KC, Trees, unicyclic and bicyclic graphs extremal with respect to multiplicative sum Zagreb index. MATCH Commun. Math. Comput. Chem. 68(1), 257 - 272(2012).
[11] K.Pattabiraman, S.Nagarajan, M.Chendrasekharan, Zagreb indices and coindices of product graphs, Journal of Prime Research in Mathematics Vol 10, 80 - 91(2015).