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η-Ricci Soliton On A Real Hypersurface Of A Complex Space Form

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Abstract

The object of the present paper is to study η- Ricci solitons on a real hypersurface of a complex space form with the semi-symmetric conditions.

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1. INTRODUCTION

An n- dimensional K$\ddot{a}$ehler manifold M^n of constant holomorphic sectional curvature c called a complex space form is either a

- complex projective space $\mathbb{C}P^n$ (for $c > 0$) or a
- complex hyperbolic space $\mathbb{C}H^n$ (for $c < 0$) or a
- Euclidean space $\mathbb{C}^n$ (for $c = 0$).

The first two forms are called non flat complex space forms and are denoted by $M^n(c)$. If M is a real hypersurface of $M^n(c)$, then the Kaehler metric G and the complex structure J on $M^n(c)$ induce an almost contact metric structure (ϕ, ξ, η, g) on M. If the structure vector field ξ is a principal vector field, i.e., if $A\xi = \alpha\xi$, where A is the shape operator of M and $\alpha = g(A\xi, \xi)$ then M is called a Hopf hypersurface and α is called the principal curvature of M.
A Ricci soliton on a Riemannian manifold generalizes the notion of Einstein metric on Riemannian manifold that has great importance in physics. Hamilton [6] was the first to introduce this notion of Ricci solitons on Riemannian manifolds.

It is well known that, if the potential vector field is zero or Killing then the Ricci soliton is an Einstein metric. In [7], [11] and [12], the authors proved that there are no Einstein real hypersurfaces of non-flat complex space forms. Motivated by this the authors Cho and Kimura [9] introduced the notion of η- Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η- Ricci solitons. Here in this paper we study totally η- umbilical real hypersurfaces of complex space forms based on associated functions of totally η- umbilical real hypersurfaces.

2. PRELIMINARIES

Let $M^n(c)$ be the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by J the complex structure and by G the Hermitian metric of $M^n(c)$.
Let M be a real $(2n-1)$- dimensional hypersurface immersed in $M^n(c)$, with the Riemannnian metric g induced from G. We take N as the unit normal vector field of M in $M^n(c)$.
For any vector field X tangent to M, we define

$$JX = \phi X + \eta(X)N, JN = -\xi \tag{2.1}$$

with the tangential part ϕX and the normal part $\eta(X)N$, where ϕ is a tensor field of type $(1,1)$, η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), g(X,\xi) = \eta(X). \tag{2.3}$$

Clearly (ϕ, ξ, η, g) defines an almost contact metric structure on M.
For an almost contact metric structure on M, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \nabla_X \xi = \phi AX. \tag{2.4}$$

We denote by R the Riemannian curvature tensor field of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y)Z = \; & c(g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z) \\ & + g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{2.5}$$

and the equation of Codazzi by

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi. \tag{2.6}$$

We call M as totally η-umbilical if the shape operator A of M is of the form

$$AX = aX + b\eta(X)\xi, \tag{2.7}$$

where a and b are some functions.
An η- Ricci soliton is a pair (η, g), which satisfies the following relation

$$L_\xi g(X,Y) + 2S(X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y) = 0, \tag{2.8}$$

where λ and μ are constants.
Throughout this paper, M denote real hypersurface of a complex space form $M^n(c)$.

3. η -RICCI SOLITONS ON REAL HYPERSURFACE OF COMPLEX SPACEFORM

Suppose M is totally η - umbilical. Then taking $X = \xi$ in (2.5) and using (2.7), we obtain

$$R(\xi,Y)Z = [c + a\alpha](g(Y,Z)\xi - \eta(Z)Y). \tag{3.1}$$

From (2.7) in (2.4), we get

$$\nabla_X \xi = \alpha \phi X. \tag{3.2}$$

It is well known that if M is a totally η- umbilical real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, $n \neq 2$, then M has two constant principal curvatures. When M is totally η - umbilical, then the sectional curvature $K(X,\xi) = g(R(X,\xi)\xi, X) = c + a^2 + ab - b^2$ is constant for any vector X orthogonal to ξ and any totally η-umbilical real hypersurface is a Hopf hypersurface. Using (2.8) and (3.2), we obtain

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{3.3}$$

In particular,

$$S(X,\xi) = S(\xi, X) = -(\lambda + \mu)\eta(X). \tag{3.4}$$

In this case, the Ricci operator Q defined by $g(QX,Y) = S(X,Y)$ has the expression

$$QX = -\lambda X - \mu\eta(X)\xi. \tag{3.5}$$

Then

$$Q\xi = -(\lambda + \mu)\xi. \tag{3.6}$$

Remark: on a Hopfhypersurface of a complex space form, the existence of an η– RicciSolitonimplies that the characteristic vector field ξ is an eigenvector of the Ricci operatorcorresponding to the eigenvalue $-(\lambda + \mu)$.

4. η -RICCI SOLITONS ON REAL HYPERSURFACE SATISFYING SEMI-SYMMETRIC CONDITIONS

Suppose $R \cdot S = 0$ holds in M. Then

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0, \tag{4.1}$$

for arbitrary vector fields X, Y, Z.

Using (3.3) in (4.1), we get

$$-\lambda g(R(\xi,X)Y,Z) - \mu\eta(R(\xi,X)Y)\eta(Z) = 0 \tag{4.2}$$

And from (3.1), (4.2) becomes

$$\mu[c + a\alpha]\{(g(X,Y) - \eta(X)\eta(Y))\eta(Z) + g(X,Z)\eta(Y) - \eta(X)\eta(Z))\eta(Y)\} = 0, \tag{4.3}$$

for arbitrary vector fields X, Y, Z. For $Z = \xi$, we have

$$\mu[c + a\alpha]g(\phi X, \phi Y) = 0, \tag{4.4}$$

for arbitrary vector fields X, Y.

From (4.4), it follows that $c = -a\alpha = -(a^2 + ab)$. Thus we can state that

Theorem 4.1: *Let M be a totally η- umbilical real hypersurface of a complex space form $M^n(c)$ admitting η-Ricci soliton. If the associated functions a and b are such that*

(i) a and b are of same sign, then the complex space form $M^n(c)$ is $\mathbb{C}H^n$.

(ii) a and b are of opposite sign, then complex space form $M^n(c)$ is $\mathbb{C}H^n$ for $a > b$ and is $\mathbb{C}P^n$ for $a < b$.

The W_2- curvature tensor introduced by Pokhariyal and Mishra[2] is given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2(n-1)}[g(X,Y)QY - g(Y,Z)QX]. \tag{4.5}$$

Suppose $W_2 \cdot S = 0$. Then

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0, \tag{4.6}$$

for arbitrary vector fields X, Y, Z.

Using (3.3) in (4.6), we get

$$-\lambda g(W_2(\xi, X)Y, Z) - \mu\eta(W_2(\xi, X)Y)\eta(Z) = 0. \tag{4.7}$$

By using (3.1), (3.5) and (3.6), we obtain

$$\begin{aligned}
&\lambda[(c + a\alpha)(g(X, Y)\eta(Z) - g(X, Z)\eta(Y) + g(X, Z)\eta(Y) - g(X, Y)\eta(Z))] \\
&+\lambda\left(\frac{1}{2(n-1)}[-\lambda(g(X, Z)\eta(Y) + g(X, Y)\eta(Z)) - 2\mu\eta(X)\eta(Y)\eta(Z)]\right) \\
&+\lambda\left(\frac{1}{2(n-1)}[(\lambda + \mu)(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))]\right) \\
&+\mu[(c + a\alpha)(g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z) + g(X, Y)\eta(Z))] \\
&+\mu\left(\frac{1}{2(n-1)}[-2\lambda\eta(X)\eta(Y)\eta(Z) - 2\mu\eta(X)\eta(Y)\eta(Z)]\right) \\
&+\mu\left(\frac{1}{2(n-1)}[(\lambda + \mu)(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))]\right) = 0,
\end{aligned} \tag{4.8}$$

for arbitrary vector fields X, Y, Z.

Taking $Z = \xi$ in (4.8), we get

$$2(n-1)\mu(c + a\alpha) + (\lambda + \mu)^2 g(\phi X, \phi Y) = 0, \tag{4.9}$$

for arbitrary vector fields X, Y.

From (4.9), it follows that

$$2(n-1)\mu(c + a\alpha) + (\lambda + \mu)^2 = 0. \tag{4.10}$$

Hence we can state

Theorem 4.2: *If (g, ξ, λ, μ) is an η- Ricci soliton on a real hypersurface M of $M^n(c)$ and $W_2(\xi, X).S = 0$, then $2(n-1)\mu(c + a\alpha) + (\lambda + \mu)^2 = 0$ holds.*

Now from (4.10), it follows that for $\mu = 0$, we get $\lambda = 0$

So we conclude that

Corollary 4.1: *If (g, ξ, λ, μ) is a Ricci soliton on a real hypersurface M of $M^n(c)$ where $W_2(\xi, X).S = 0$ is steady.*

The conhormonic curvature tensor $\tilde{C}$ in M is defined by [10]

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{2n-3}(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY) \tag{4.11}$$

Suppose $\tilde{C} \cdot S = 0$ holds in M. Then

$$S(\tilde{C}(\xi, X)Y, Z) + S(Y, \tilde{C}(\xi, X)Z) = 0, \tag{4.12}$$

for arbitrary vector fields X, Y, Z.

Replacing the expression of S from (3.3), we get

$$-\lambda g(\tilde{C}(\xi, X)Y, Z) - \mu\eta(\tilde{C}(\xi, X)Y)\eta(Z) = 0. \tag{4.13}$$

Using (3.1),(3.3) and (3.4),we obtain

$$\begin{aligned}
&\mu[(c + a\alpha)(g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z))] \\
&-\frac{\mu}{(2n-3)}[-2\lambda(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))] \\
&+ - \frac{\mu}{(2n-3)}[\mu(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))] \\
&-\frac{\mu}{(2n-3)}[4\lambda\eta(X)\eta(Y)\eta(Z) + 2\mu\eta(X)\eta(Y)\eta(Z)] = 0,
\end{aligned} \tag{4.14}$$

for arbitrary vector fields X, Y, Z.

Taking $Z = \xi$ in (4.14), we get
$$\mu[(2n-3)[c+a\alpha]+(2\lambda+\mu)]g(\phi X,\phi Y)=0, \tag{4.15}$$
for arbitrary vector fields X, Y.
From (4.15), it follows that $[(2n-3)[c+a\alpha]+(2\lambda+\mu)]=0$.
Hence we can state

Theorem 4.3: *If* (g,ξ,λ,μ) *is an* η*- Ricci soliton on a real hypersurface* M *of* $M^n(c)$ *and* $\tilde{C}(\xi,X).S=0$, *then* $c=-\left(a\alpha+\frac{(2\lambda+\mu)}{(2n-3)}\right)$.

The projective curvature tensor P in M is defined by [8]
$$P(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}(S(Y,Z)X-S(X,Z)Y) \tag{4.16}$$
Suppose $P\cdot S=0$. Then
$$S(P(\xi,X)Y,Z)+S(Y,P(\xi,X)Z)=0, \tag{4.17}$$
for arbitrary vector fields X, Y, Z.
From (3.3) in (4.17), we get
$$-\lambda g(P(\xi,X)Y,Z)-\mu\eta(P(\xi,X)Y)\eta(Z)=0. \tag{4.18}$$
Using (3.1), (3.3) and (3.4), we get
$$\begin{aligned}
&\lambda[(c+a\alpha)(g(X,Y)\eta(Z)-g(X,Z)\eta(Y)+g(X,Z)\eta(Y)-g(X,Y)\eta(Z))]\\
&+\lambda\left(\frac{1}{(n-1)}[-\lambda(g(X,Y)\eta(Z)+g(X,Z)\eta(Y))-2\mu\eta(X)\eta(Y)\eta(Z)]\right)\\
&+\lambda\left(\frac{1}{(n-1)}[(\lambda+\mu)(g(X,Z)\eta(Y)+g(X,Y)\eta(Z))]\right)\\
&+\mu[(c+a\alpha)(g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(X,Z)\eta(Y))]\\
&+\mu\left(\frac{1}{(n-1)}[-\lambda(g(X,Y)\eta(Z)+g(X,Z)\eta(Y))-2\mu\eta(X)\eta(Y)\eta(Z)]\right)\\
&+\mu\left(\frac{1}{(n-1)}[2(\lambda+\mu)\eta(X)\eta(Y)\eta(Z)]\right)=0,
\end{aligned} \tag{4.19}$$
for arbitrary vector fields X, Y, Z.
Putting $Z=\xi$ in the above, we get
$$\mu((n-1)[c+a\alpha]+\lambda)g(\phi X,\phi Y)=0, \tag{4.20}$$
for arbitrary vector fields X, Y.
In (4.20) $\mu\neq 0$ then, it follows that $((n-1)[c+a\alpha]+\lambda)=0$.
Hence we can state.

Theorem 4.4: *If* (g,ξ,λ,μ) *is an* η*- Ricci soliton on a real hypersurface* M *of* $M^n(c)$ *and* $P(\xi,X).S=0$, *then* $c=-\left(a\alpha+\frac{\lambda}{n-1}\right)$.

Suppose that M is projectively flat,that is, $P(X,Y)Z=0$ for all vector fields X, Y, Z.
Then from (4.16), we obtain
$$R(X,Y)Z=\frac{1}{n-1}(S(Y,Z)X-S(X,Z)Y) \tag{4.21}$$
and
$$R(\xi,Y)Z=\frac{1}{n-1}(S(Y,Z)\xi-S(\xi,Z)Y). \tag{4.22}$$
We consider
$$(R(\xi,X).S)(Y,Z)=S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z). \tag{4.23}$$
Applying (3.3) in (4.23), we get

$$
\begin{aligned}
&\frac{\lambda}{n-1}[-\lambda(g(X,Y)\eta(Z)+g(X,Z)\eta(Y))-2\mu\eta(X)\eta(Y)\eta(Z)]\\
&+\frac{\lambda}{n-1}[(\lambda+\mu)(g(X,Z)\eta(Y)+g(X,Y)\eta(Z))]\\
&+\frac{\mu}{n-1}[-\lambda(g(X,Y)\eta(Z)+g(X,Z)\eta(Y))-2\mu\eta(X)\eta(Y)\eta(Z)]\\
&+\frac{\mu}{n-1}[2(\lambda+\mu)\eta(X)\eta(Y)\eta(Z)]
\end{aligned} \tag{4.24}
$$

for arbitrary vector fields X, Y, Z. Putting $Z = \xi$ in (4.24), we get

$$
\begin{aligned}
&\frac{\lambda}{n-1}(\lambda(\eta(X)\eta(Y)-g(X,Y)+g(X,Y)-\eta(X)\eta(Y))\\
&+\frac{\mu}{n-1}(\lambda(\eta(X)\eta(Y)-g(X,Y)+g(X,Y)-\eta(X)\eta(Y))=0.
\end{aligned} \tag{4.25}
$$

Therefore

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0 \tag{4.26}$$

or

$$R(\xi,X)\cdot S=0. \tag{4.27}$$

Combining this with theorem(4.1), we get $c=-a\alpha=-(a^2+ab)$.
Thus we can state that

Theorem 4.5:*Let M be a projectively flat totally η- umbilical real hypersurface of a complex space form $M^n(c)$ admitting η-Ricci soliton. If the associated functions a and b are such that*

(i) a and b are of same sign, then the complex space form $M^n(c)$ is $\mathbb{C}H^n$.

(ii) a and b are of opposite sign, then complex space form $M^n(c)$ is $\mathbb{C}H^n$ for $a > b$ and is $\mathbb{C}P^n$ for $a < b$.

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