Super Geometric Mean Labeling Of Some Disconnected Graphs

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ABSTRACT

Let f: V(G) → {1,2,...,p+q} be an injective function. For a vertex labeling “f”, the induced edge labeling f*(e=uv) is defined by, f*(e)=[(f(u)+f(v))/2] or [(f(u)f(v))]. Then “f” is called a “Super Geometric mean labeling” if \{f(V(G))\}∪{f(e):e∈E(G)}={1,2,...,p+q}. A graph which admits Super Geometric mean labeling is called “Super Geometric mean graph”. In this paper we prove that some disconnected graphs are Super Geometric mean graphs.

Key words: Graph, Super Geometric mean graph, Path, Comb and Ladder.
1. Introduction
The graphs considered here are simple, finite and undirected graphs. Let $V(G)$ denote the vertex set of $G$ and $E(G)$ denote the edge set of $G$. For a detailed survey of graph labeling we refer to Gallian [1]. For all other standard terminology and notations we follow Harary [2]. The concept of “Geometric mean labeling” has been introduced by S.Somasundaram, R.Ponraj and P.Vidhyarani in [4]. S.S.Sandhya, E. Ebin Raja Merly and B.Shiny introduced “Super Geometric mean labeling” in [5].
In this paper, we investigate “Super Geometric mean labeling” behavior of some disconnected graphs.
Now we will give the following definitions which are necessary for our present investigation.

Definition: 1.1
A graph $G=(V,E)$ with $p$ vertices and $q$ edges is called a “Geometric mean graph” if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \ldots, q+1$ in such a way that when each edge $e=uv$ is labeled with $f(e=uv)=\left[\sqrt[2]{f(u)f(v)}\right]$ or $\left[\sqrt[3]{f(u)f(v)}\right]$ then the edge labels are distinct. In this case, “$f$” is called a “Geometric mean labeling” of $G$.

Definition: 1.2
Let $f: V(G) \rightarrow \{1, 2, \ldots, p+q\}$ be an injective function. For a vertex labeling “$f$”, the induced edge labeling $f^* (e=uv)$ is defined by,
$$f^*(e)=\left[\sqrt[2]{f(u)f(v)}\right] \text{ or } \left[\sqrt[3]{f(u)f(v)}\right].$$
Then “$f$” is called a “Super Geometric mean labeling” if \{f(V(G)))\} $\cup$ \{f(e):e $\in$ E(G)$\}=\{1, 2, \ldots, p+q\}$. A graph which admits Super Geometric mean labeling is called “Super Geometric mean graph”.

Definition: 1.3
The union of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2, E_2)$ is a graph $G=G_1 \cup G_2$ with vertex set $V=V_1 \cup V_2$ and the edge set $E=E_1 \cup E_2$.

Definition: 1.4
A path $P_n$ is a walk in which all the vertices are distinct.

Definition: 1.5
A graph obtained by joining a single pendant edge to each vertex of a path is called a Comb ($P_n \vee K_1$).

Definition: 1.6
The Ladder $L_n$, $n \geq 2$ is the product graph $P_n \times P_2$ and contains $2n$ vertices and $3n-2$ edges.

Definition: 1.7
The graph $P_n \vee K_{1,2}$ is obtained by attaching $K_{1,2}$ to each vertex of $P_n$. 
**Definition:** 1.8  
The graph $P_n \triangle K_{1,3}$ is obtained by attaching $K_{1,3}$ to each vertex of $P_n$.

2. Main Results  
**Theorem:** 2.1  
$P_m \cup P_n$ is a Super Geometric mean graph.  

**Proof:**  
Let $P_m = u_1 u_2 \ldots u_m$ be a path on “m” vertices.  
Let $P_n = t_1 t_2 \ldots t_n$ be another one path on “n” vertices.  
Let $G = P_m \cup P_n$  
Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by,  

$f(u_i) = 2i - 1, 1 \leq i \leq m$  
$f(t_i) = 2m + 2i - 2, 1 \leq i \leq n$  

Edge labels are given by,  

$f(u_i u_{i+1}) = 2i, 1 \leq i \leq m - 1$  
$f(t_i t_{i+1}) = 2m + 2i - 1, 1 \leq i \leq n - 1$  

: The edge labels are distinct.  
Thus “f” provides a Super Geometric mean labeling.  
Hence $P_m \cup P_n$ is a Super Geometric mean graph.

**Example:** 2.2  
A Super Geometric mean labeling of $P_7 \cup P_8$ is shown below.

![Figure: 1](image)

**Theorem:** 2.3  
$(P_m \triangle K_1) \cup P_n$ is a Super Geometric mean graph.  

**Proof:**  
Let $(P_m \triangle K_1)$ be a Comb graph obtained from a path $P_m = v_1 v_2 \ldots v_m$ by joining a vertex $u_i$ to $v_i$, $1 \leq i \leq m$. Let $P_n = w_1 w_2 \ldots w_n$ be a path.  
Let $G = (P_m \triangle K_1) \cup P_n$  
Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by,  

$f(v_i) = 4i - 1, 1 \leq i \leq m$  
$f(w_i) = 2m + 2i - 2, 1 \leq i \leq n$  

Edge labels are given by,  

$f(u_i u_{i+1}) = 2i, 1 \leq i \leq m - 1$  
$f(t_i t_{i+1}) = 2m + 2i - 1, 1 \leq i \leq n - 1$  

: The edge labels are distinct.  
Thus “f” provides a Super Geometric mean labeling.  
Hence $(P_m \triangle K_1) \cup P_n$ is a Super Geometric mean graph.
\( f(u_i) = 4i-3, \quad 1 \leq i \leq m \)
\( f(w_i) = 4m+2i-2, \quad 1 \leq i \leq n \)

Edges are labeled with,
\( f(v_i v_{i+1}) = 4i, \quad 1 \leq i \leq m-1 \)
\( f(u_i v_i) = 4i-2, \quad 1 \leq i \leq m \)
\( f(w_i w_{i+1}) = 4m+2i-1, \quad 1 \leq i \leq n-1 \)

Thus we get distinct edge labels.

Hence \((P_m \cup K_1) \cup P_n\) is a Super Geometric mean graph.

**Example: 2.4**

Super Geometric mean labeling of \((P_6 \cup K_1) \cup P_5\) is given below.

\[
\begin{align*}
G &= L_m \cup P_n.
\end{align*}
\]

\[
\begin{align*}
f(v_1) &= 1 \\
f(v_i) &= 5i-2, \quad 2 \leq i \leq m. \\
f(u_1) &= 4 \\
f(u_i) &= 5i-4, \quad 2 \leq i \leq m. \\
f(w_1) &= 5m+2i-3, \quad 1 \leq i \leq n. \\
\end{align*}
\]

Edges are labeled with,
\( f(v_1 v_2) = 3 \)
\( f(v_i v_{i+1}) = 5i, \quad 2 \leq i \leq m-1 \)
\( f(u_1 u_2) = 5, \)

**Theorem: 2.5**

\( L_m \cup P_n \) is a Super Geometric mean graph.

**Proof:**

Let \( L_m = P_m \times P_2 \) be a ladder, \( P_m = v_1 v_2 \ldots v_m \) Let \( P_n = w_1 w_2 \ldots w_n \) be a path

Let \( G = L_m \cup P_n \).

Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by,
\( f(v_1) = 1 \)
\( f(v_i) = 5i-2, \quad 2 \leq i \leq m. \)
\( f(u_1) = 4 \)
\( f(u_i) = 5i-4, \quad 2 \leq i \leq m. \)
\( f(w_1) = 5m+2i-3, \quad 1 \leq i \leq n. \)

Edges are labeled with,
\( f(v_1 v_2) = 3 \)
\( f(v_i v_{i+1}) = 5i, \quad 2 \leq i \leq m-1 \)
\( f(u_1 u_2) = 5, \)
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\[ f(u_i u_{i+1}) = 5i - 1, \ 2 \leq i \leq m - 1 \]
\[ f(v_i u_i) = 5i - 3, \ 1 \leq i \leq m \]
\[ f(w_i w_{i+1}) = 5m + 2i - 2, \ 1 \leq i \leq n - 1 \]
\[ \therefore \text{We get distinct edge labels.} \]
\[ \therefore \text{Hence "f" provides a Super Geometric mean labeling.} \]
\[ \therefore L_m \cup P_n \text{ is a Super Geometric mean graph.} \]

**Example: 2.6**

Super Geometric mean labeling of \( L_5 \cup P_6 \) is displayed below.

![Figure: 3](image)

**Theorem: 2.7**

\( (P_m \cup K_{1,2}) \cup P_n \) is a Super Geometric mean graph.

**Proof:**

Let \( (P_m \cup K_{1,2}) \) be a graph obtained by attaching each vertex of a path \( P_m \) to the central vertex of \( K_{1,2} \) where \( P_m = u_1 u_2 \ldots u_m \).

Let \( v_i \) and \( w_i \) be the vertices of \( K_{1,2} \) which are attached with the vertex \( u_i \) of \( P_m \), \( 1 \leq i \leq m \).

Let \( P_m = z_1 z_2 \ldots z_n \) be a path.

Let \( G = (P_m \cup K_{1,2}) \cup P_n \).

Define a function \( f: V(G) \to \{1,2,\ldots, p+q\} \) by

\[ f(u_i) = 6i - 3, \ 1 \leq i \leq m \]
\[ f(v_i) = 6i - 5, \ 1 \leq i \leq m \]
\[ f(w_i) = 6i - 1, \ 1 \leq i \leq m \]
\[ f(z_i) = 6m + 2i - 2, \ 1 \leq i \leq n \]

Edges are labeled with,

\[ f(u_i u_{i+1}) = 6i, \ 1 \leq i \leq m - 1 \]
\[ f(u_i v_i) = 6i - 4, \ 1 \leq i \leq m \]
f(u_iw_i)=6i-2, 1≤i≤m
f(z_iz_{i+1})=6m+2i-1, 1≤i≤n-1

\therefore The edge labels are distinct.
Hence G admits a Super Geometric mean labeling.
Hence (P_m\Delta K_{1,2})\cup P_n is a Super Geometric mean graph.

**Example: 2.8**
Super Geometric mean labeling of (P_4\Delta K_{1,2})\cup P_5 is shown below.

![Figure 4](image)

**Theorem: 2.9**
(P_n\Delta K_{1,3}) is a Super Geometric mean graph.

**Proof:**
Let (P_n\Delta K_{1,3}) be a graph obtained by attaching each vertex of a path P_m=u_1u_2...u_m to the central vertex of K_{1,3}.
Let v_i, w_i and z_i be the vertices of K_{1,3} which are attached with the vertex u_i of P_m, 1≤i≤m.
Let P_n=t_1t_2...t_n be a path.
Let G=(P_m\Delta K_{1,3})\cup P_n
Define a function f: V(G)→\{1,2,...,p+q\} by,
f(u_i)=8i-3, 1≤i≤m
f(v_i)=8i-7, 1≤i≤m
f(w_i)=8i-5, 1≤i≤m
f(z_i)=8i-1, 1≤i≤m
f(t_i)=8m+2i-2, 1≤i≤n
Edges are labeled with,
f(u_iu_{i+1})=8i, 1≤i≤m-1
f(u_iv_i)=8i-6, 1≤i≤m
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\[ f(u_iw_i) = 8i - 4, \quad 1 \leq i \leq m \]
\[ f(u_iz_i) = 8i - 2, \quad 1 \leq i \leq m \]
\[ f(t_{i+1}) = 8m + 2i - 1, \quad 1 \leq i \leq n - 1 \]

From the above labeling pattern, both vertices and edges together get distinct labels from \( \{1, 2, \ldots, p+q\} \).

Hence \((P_m \bigcirc K_{1,3}) \bigcirc P_n\) is a Super Geometric mean graph.

**Example: 2.10**

Super Geometric mean labeling of \((P_5 \bigcirc K_{1,3}) \bigcirc P_4\) is given below.

![Graph Image]

*Figure: 5*

**Theorem: 2.11**

Let \( G_1 \) be a graph obtained from a path \( P_m = v_1v_2 \ldots v_m \) by joining pendant vertices with the vertices of the path \( P_m \) alternatively. Let \( P_n = w_1w_2 \ldots w_n \) be another path. Let \( G = G_1 \bigcirc P_n \). Then \( G \) is a Super Geometric mean graph.

**Proof:**

Let \( G_1 \) be a graph obtained from a path \( P_m = v_1v_2 \ldots v_m \) by joining pendant vertices with the vertices of the path \( P_m \), alternatively.

Let \( P_n = w_1w_2 \ldots w_n \) be another one path.

Let \( G = G_1 \bigcirc P_n \)

Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by,

\[ f(v_i) = 3i, \quad i = 1, 3, 5, \ldots, m \]

\[ f(v_{2i}) = 6i - 1, \quad 1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor \]

\[ f(u_i) = 3i - 2, \quad i = 1, 3, 5, \ldots, m \]

\[ f(w_i) = 3m + 2i - 1, \quad 1 \leq i \leq n \]

Edges are labeled with,

\[ f(v_{ij}v_{i+1}) = 3i + 1, \quad i = 1, 3, 5, \ldots, m - 2 \]

\[ f(v_{2i}v_{2i+1}) = 6i, \quad 1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor \]

\[ f(v_{ij}u_i) = 3i - 1, \quad i = 1, 3, 5, \ldots, m \]
f(w_{i+1})=3m+2i, 1 \leq i \leq n-1
\therefore \text{We get distinct edge labels.}
\text{Hence } \{f(V(G))\} \cup \{f(e): e \in E(G)\} = \{1, 2, \ldots, p+q\}.
\text{Hence } G \text{ is a Super Geometric mean graph.}

\textbf{Example: 2.12}
Let $G_1$ be a graph obtained from a path $P_9$ by joining pendant vertices with the vertices of $P_9$ alternatively. A Super Geometric mean labeling of $G = G_1 \cup P_3$ is displayed below.

\textbf{Theorem: 2.13}
Let $G_1$ be a graph obtained from a Ladder $L_m$, $m \geq 2$ by joining a pendant vertex with a vertex of degree two on both sides of upper and lower path of the ladder. Let $P_n=t_1t_2\ldots t_n$ be another path. Let $G = G_1 \cup P_n$. Then $G$ is a Super Geometric mean graph.

\textbf{Proof:}
Let $L_m=P_n \times P_2$ be a Ladder graph.
Let $G_1$ be a graph obtained from a Ladder by joining pendant vertices $u, w, x, z$ with $v_1, v_n, u_1, u_n$ (vertices of degree 2) respectively on both sides of upper and lower path of the ladder.
Let $P_n=t_1t_2\ldots t_n$ be another one path.
Let $G = G_1 \cup P_n$
Define a function $f: V(G) \rightarrow \{1, 2, \ldots, p+q\}$ by,
$f(u)=1$
$f(v_1)=5$
$f(v_i)=5i-1, \ 2 \leq i \leq n$
$f(w)=5m+5$
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\[ f(x) = 3 \]
\[ f(u_i) = 5i + 3, \quad 1 \leq i \leq m \]
\[ f(z) = 5m + 6 \]
\[ f(t_i) = 5m + 2i + 5, \quad 1 \leq i \leq n \]

Edges are labeled with,
\[ f(v_i v_{i+1}) = 5i + 2, \quad 1 \leq i \leq m - 1 \]
\[ f(u v_1) = 2 \]
\[ f(v_m w) = 5m + 2 \]
\[ f(x u_1) = 4 \]
\[ f(u_1 u_{i+1}) = 5i + 5, \quad 1 \leq i \leq m - 1 \]
\[ f(u_m z) = 5m + 4 \]
\[ f(v_i u_i) = 5i + 1, \quad 1 \leq i \leq m \]
\[ f(t_{i+1}) = 5m + 2i + 6, \quad 1 \leq i \leq n - 1 \]

In view of the above labeling pattern, \( f \) provides a Super Geometric mean labeling of \( G \).
Hence \( G \) is Super Geometric mean graph.

**Example: 2.14**

A super Geometric mean labeling of \( G \) when \( m = 5 \) and \( n = 6 \) is shown below.

![Figure: 7](image)

**Theorem: 2.15**

Let \( G_1 \) be a graph obtained by joining a pendant vertex with a vertex of degree two on both sides of a Comb graph. Let \( P_n = w_1 w_2 \ldots w_n \) be another path. Let \( G = G_1 \cup P_n \). Then \( G \) is a Super Geometric mean graph.

**Proof:**

Comb \( (P_m \Delta K_1) \) is a graph obtained from a path \( P_m = v_1 v_2 \ldots v_m \) by joining a vertex \( u_i \) to \( v_i, \quad 1 \leq i \leq m. \)

Let \( G_1 \) be a graph obtained by joining pendant vertices \( w \) and \( z \) to \( v_1 \) and \( v_m \) respectively.

Let \( P_n = w_1 w_2 \ldots w_n \) be another one path.
Let $G=G_1 \cup P_n$.
Define a function $f: V(G) \rightarrow \{1, 2, \ldots, p+q\}$ by,
\[
  f(w) = 1 \\
  f(v_1) = 3 \\
  f(v_i) = 4i+1, \ 2 \leq i \leq m \\
  f(z) = 4m+3 \\
  f(u_1) = 5 \\
  f(u_i) = 4i-1, \ 2 \leq i \leq m \\
  f(w_i) = 4m+2i+2, \ 1 \leq i \leq n \\
\]
Edges are labeled with
\[
  f(wv_1) = 2 \\
  f(v_iv_{i+1}) = 4i+2, \ 1 \leq i \leq m-1 \\
  f(v_nz) = 4m+2 \\
  f(v_iu_i) = 4i, \ 1 \leq i \leq m \\
  f(w_iw_{i+1}) = 4m+2i+3, \ 1 \leq i \leq n-1 \\
\]
\[
  \therefore \text{The edge labels are distinct.} \\
  \text{Hence} \ G \text{ is a Super Geometric mean graph.}
\]

**Example: 2.16**
A Super Geometric mean labeling of $G$ when $m=5$, and $n=4$ is displayed below.

**Theorem 2.17**
Let $P_m$ be a path and $G_1$ be the graph obtained from $P_m$ by attaching $C_3$ in both end edges of $P_m$. Let $P_n=w_1w_2\ldots w_n$ be another path. Let $G=G_1 \cup P_n$. Then $G$ is a Super Geometric mean graph.

**Proof:**
Let $P_m$ be a path $u_1u_2\ldots u_n$ and $v_1u_1u_2$, $v_2u_{n-1}u_n$ be the triangles at the end edges of $P_m$. The resulting graph is $G_1$. 

![Diagram of Super Geometric mean labeling](image-url)
Let $P_n = w_1w_2...w_n$ be another one path.
Let $G = G_1 \cup P_n$.
Define a function $f: V(G) \rightarrow \{1, 2, ..., p+q\}$ by,
\[
\begin{align*}
    f(v_1) &= 4 \\
    f(u_1) &= 1 \\
    f(u_i) &= 2i+2, \ 2 \leq i \leq m-1 \\
    f(u_m) &= 2m+5 \\
    f(v_2) &= 2m+2 \\
    f(w_i) &= 2m+2i+4, \ 1 \leq i \leq n \\
\end{align*}
\]
Edges are labeled with
\[
\begin{align*}
    f(v_1u_1) &= 2 \\
    f(v_1u_2) &= 5 \\
    f(u_1u_2) &= 3 \\
    f(u_iu_{i+1}) &= 2i+3, \ 2 \leq i \leq m-2 \\
    f(u_{m-1}u_m) &= 2m+3 \\
    f(v_2u_{m-1}) &= 2m+1 \\
    f(v_2u_m) &= 2m+4 \\
    f(w_iw_{i+1}) &= 2m+2i+5, \ 1 \leq i \leq n-1 \\
\end{align*}
\]
\[.\] We get distinct edge labels.
Thus both vertices and edges together get distinct labels from \{1, 2, ..., p+q\}.
Hence $G$ is a Super Geometric mean graph

**Example: 2.18**
A Super Geometric mean labeling of $G$ when $m=8$ and $n=5$, is shown below.
References: