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An Extension Of Gregus Fixed Point Theorem

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Abstract

Let C be a closed convex subset of a complete metrizable topological vector space (X, d) and T: $C \rightarrow C$ a mapping that satisfies d (Tx, Ty) $\leq$ ad (x, y) + bd (x, Tx) + cd (y, Ty) + ed (y, Tx) + fd (x, Ty) for all x, $y \in C$, where 0<a<1, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$, and a + b + c + e + f = 1. Then T has a unique fixed point. The above theorem, which is a generalization and an extension of the results of several others, is proved in this paper. In addition, we use the Ishikawa iteration to approximate the fixed point of T.

Key words: complete metrizable topological vector space, common fixed point, Ishikawa iteration.

AMS subject classification (2000): 54H25, 47H10.

Introduction:

Gregus [1] proved the following theorem.

Theorem 1. 1 Let C be a closed convex subset of a Banach space X and T: $C \rightarrow C$ a mapping that satisfies $\|Tx - Ty\| \leq a \|x - y\| + b \|x - Tx\| + c \|y - Ty\|$ for all x, $y \in C$, where 0<a<1, $b \geq 0$, $c \geq 0$ and a + b + c = 1. Then T has a unique fixed point.
Several papers have been written on the Gregus fixed point theorem. For example, see [2 - 3]. The theorem has been generalized to the condition when X is a complete matrizable topological vector space[4].
In this paper, we extend Gregus result to the condition when T satisfies the condition given in Theorem1. 1 and also generalize it to the condition when X is a complete metrizable topological vector space. The following result will be needed for our result.

Theorem 1. 2: A topological vector space X is matrizable iff it has a countable base of neighbourhoods of zero. The topology of a metrizable topological vector space can always be defined by a real - valued function $\|.\|: X \to R$, called F - norm such that for all $x, y \in X$,

(1) $\| x \| \geq 0$

(2) $\| x \| = 0 \Rightarrow x = 0$

(3) $\| x + y \| \leq \|x\| + \|y\|$

(4) $\| \lambda x \| \leq \|x\|$ forall $\lambda \in K$ with $|\lambda| \leq 1$

(5) if $\lambda_n \to 0$ and $\lambda_n \in K$, then $\| \lambda_n x \| \to 0$

Theorem 1. 3: Let C be a closed convex subset of a complete matrizable space X and $T: C \to C$ a mapping that satisfies $F(Tx - Ty) \leq aF(x - y) + bF(x - Tx) + cF(y - Ty) + eF(y - Tx) + fF(x - Ty)$ for akkx, $y \in C$, where $0<a<1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$ and $a + b + c + e + f = 1$. Then T has a unique fixed point.

Proof: Take any point $x \in C$ and consider the sequence $\{T_n(x)\}_{n=1}^{\infty}$

$$F(T^n x - T^{n-1}x) \leq aF(T^{n-1}x - T^{n-2}x) + bF(T^{n-1}x - T^n x) + cF(T^{n-2}x - T^{n-1}x) + eF(T^{n-2}x - T^{n-1}x) + eF(T^{n-2}x - T^n x) + fF(T^{n-1}x - T^{n-1}x) \quad (1.1)$$

$$\leq \frac{a+c+e}{1-b-e} F(T^{n-1}x - T^{n-2}x)$$

$$\leq \frac{a+2p}{1-2p} F(T^{n-1}x - T^{n-2}x) \leq F(Tx - x) \quad (1.2)$$

Thus $F(T^n x - T^{n-1}x) \leq F(Tx - x)$ and

$$F(T^3x - Tx) \leq aF(T^2x - x) + bF(T^2x - T^3x) + cF(Tx - x) + eF(x - T^3x) + fF(T^2x - Tx)$$

$$\leq aF(T^2x - Tx) + aF(Tx - x) + bF(T^2x - T^3x) + cF(Tx - x) + eF(x - Tx) + eF(Tx - T^2x) + eF(T^2x - T^3x) + fF(T^2x - Tx) \quad (1.3)$$

$$\leq (2a + b + c + 3e + f)F(Tx - x) = (a + 2p + 1)F(Tx - x)$$

Hence $F(T^3x - Tx) \leq (a + 2p + 1)F(Tx - x) \ \forall x \in C$ (1. 4)

Since C is convex, therefore $z = \frac{1}{2}T^2x + \frac{1}{2}T^3x$ is in C, and from the properties of F - norm, we have $F(Tz - z) \leq \frac{1}{2}F(Tz - T^2x) + \frac{1}{2}F(Tz - T^3x)$

$$\leq \frac{1}{2}[aF(z - Tx) + bF(Tz - z) + cF(Tx - T^2x) + eF(Tx - Tz) + fF(z - T^2x)]$$

$$+ \frac{1}{2}\{aF(z - T^2x) + bF(Tz - z) + cF(T^3x - T^2x) + eF(T^2x - Tz) + fF(z - T^3x)\}$$

$$F(z - Tx) \leq \frac{1}{2}F(T^2x - Tx) + \frac{1}{2}F(T^3x - Tx) \leq \frac{1}{2}F(Tx - x) + \frac{1}{2}(a + 2p + 1)F(Tx - x)$$

$$= (1 + p + \frac{1}{2}a)F(Tx - x)$$

$F(z - T^2x) \le \frac{1}{2} F(T3x - T2x) \le \frac{1}{2} F(Tx - x)$ (1. 5)

Similarly, $F(z - T^3x) \le \frac{1}{2} F(Tx - x)$, $F(Tx - Tz) \le \frac{1}{2} F(Tx - T^3x) + \frac{1}{2} F(Tx - T^4x)$

$\le \frac{1}{2}(a + 2p + 1)F(Tx - x) + \frac{1}{2}\{F(Tx - T^2x) + F(T^2x - T^4x)\}$

$\le \frac{1}{2}(a + 2p + 1)F(Tx - x) + \frac{1}{2}\{F(Tx - x) + (a + 2p + 1)F(Tx - x)\}$

$\le (a + 2p + \frac{3}{2})F(Tx - x)$

$F(Tx^2 - Tz) \le \frac{1}{2} F(T^2x - T^3x) + \frac{1}{2} F(T^2x - T^4x) \le (\frac{1}{2} a + p + 1)F(Tx - x)$ (1. 6)

Thus $(1 - b)F(Tz - z) \le \frac{1}{2}\{a(1 + p + \frac{1}{2} a)F(Tx - x) + cF(Tx - x)$

$+ eF(a + 2p + \frac{3}{2})F(Tx - x) + \frac{1}{2} fF(Tx - x)\} + \frac{1}{2}\{\frac{1}{2} aF(Tx - x)$

$+ cF(Tx - x) + \frac{1}{2} e(a + 2p + 1)F(Tx - x) + \frac{1}{2} fF(Tx - x)\}$

$= (\frac{3}{4} a + \frac{1}{4} a^2 + \frac{5}{4} ap + \frac{5}{2} p + \frac{3}{2} p^2)F(Tx - x)$ (1. 7)

Thus $4(1 - p)F(z - Tz) \le (3a + a^2 + 5ap + 10p + 6p^2)F(Tx - x)$
$\le (2p^2 - 5p + 4)F(Tx - x)$ (1. 8)
Hence

$F(z - Tz) \le \frac{26 - 22a - a2}{8(a + 3)} F(Tx - x) \le \lambda F(Tx - x)$, (1. 9)

Where $\lambda = \frac{26 - 22a - a2}{8(a + 3)}$. It is clear that $0 < \lambda < 1$.

Now let $i = \inf\{F(x - x): x \in C\}$. Then there exists a point $x \in C$ such that $F(Tx - x) < i + \varepsilon$ for $\varepsilon > 0$. Suppose $i \ge 0$. Then for $0 < \varepsilon < \frac{(1-\lambda)i}{\lambda}$, and $F(Tx - x) < i + \varepsilon$, we have

$F(Tz - z) \le \lambda F(Tx - x) \le \lambda(i + \varepsilon) < i$, (1. 10)

that is, $F(Tz - z) < i$, which is a contradiction with the definition of i. Hence $\inf\{F(Tx - x): x \in C\} = 0$.

To prove that the infimum is attained is the easy part of the proof. Take the following system of sets:

$K_n = \{x: F(x - Tx) \le \frac{1}{2n}(q + 1)\}$; $T(K_n)$ and $\overline{T(Kn)}$, where $n \in N$, $q = \frac{a + p}{1 - a}$, and $\overline{T(Kn)}$ is the closure of $T(K_n)$. Then for any $x, y \in K_n$,

$F(Tx - Ty) \le qF(Tx - x) + qF(Ty - y) \le \frac{1}{n}$,

$F(x - y) \leq (q + 1)F(Tx - x) + (q + 1)F(Ty - y) \leq \frac{1}{n}$, (1. 11)

that is, diam $(K_n) \leq \frac{1}{n}$, diam $(T(K_n)) \leq \frac{1}{n}$ and therefore, since diam $(T(K_n))$ = diam $(\overline{T(Kn)})$, we have diam $(\overline{T(Kn)}) \leq \frac{1}{n}$. It is clear that $\{K_n\}$ and $\{T(K_n)\}$ from monotone sequences of sets and from (1. 2) we have $T(K_n) \subset K_n$. Suppose $y \in \overline{T(Kn)}$, then there exists $y' \in K_n$, such that $F(y - Ty') < \varepsilon$, for $\varepsilon > 0$ and $F(y - Ty) \leq F(y - Ty') + F(Ty' - Ty) \leq F(y - Ty') + aF(y - y') + bF(y' - Ty') + cF(Ty - y) + eF(y - Ty') + fF(y' - Ty)$ (1. 12)

Hence $(1 - c)F(y - Ty) \leq (1 + a + e + f)\varepsilon + (a + b)F(Ty' - y')$ (1. 13)

Since $F(y' - Ty') \leq \frac{1}{2n}(q + 1)$, then $F(y - Ty) \leq \frac{1}{2n}(q + 1)$, and we have $y \in K_n$. Hence $\overline{T(Kn)} \subset K_n$. $\{\overline{T(Kn)}\}$ is a decreasing sequence of closed nonempty sets with diam $(\overline{T(Kn)}) \to 0$ as $n \to \infty$. Hence they have a nonempty intersection $\{x^*\}$ and T has a unique fixed point $Tx^* = x^*$.

Cor. : If e = 0 = f then it reduces to [1].

Main Result:

We now proceed to use the Ishikawa iteration scheme to approximate the fixed point of our mapping under consideration.

Theorem 1. 4: Let C be a nonempty closed convex subset of a complex metrizable topological vector space X and let T, S: $C \to C$ be a mapping that satisfies

$F(Tx - Sy) \leq aF(x - y) + bF(Tx - x) + cF(Sy - y) + eF(Tx - y) + fF(Sy - x)$ (1. 14)

for all $x, y \in C$, where $0<a<1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$ and $a + b + c + e + f = 1$. Proof: Suppose $\{x_n\}$is a Ishikawa iteration sequence defined by

$x_0 \in X$ (1. 15)

$y_{2n} = \beta_{2n} Tx_{2n} + (1 - \beta_{2n}) x_{2n}, n \geq 0$ (1. 16)

$x_{2n+1} = \alpha_{2n} Sy_{2n} + (1 - \alpha_{2n}) x_{2n}, n \geq 0$ (1. 17)

(i) In the Ishikawa scheme $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfy $0 \leq \alpha_{2n}, \beta_{2n} \leq 1$ for all n and $\sum \alpha_{2n} \beta_{2n} = \infty$ as $n \to \infty$,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha > 0$,

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta < 1$.

If $\beta_{2n} = 0$ then Ishikawa iteration process reduces to Mann iteration process. Then $\{x_n\}$ converges to the unique fixed point of T and S.

Proof: It follows from (1. 4. 3) that $x_{2n+1} - x_{2n} = \alpha_{2n}Sy_{2n} - \alpha_{2n}x_{2n} = \alpha_{2n}(Sy_{2n} - x_{2n})$. If $x_{2n} \to z$ then $F(x_{2n+1} - x_{2n}) \to 0$;Since $\{\alpha_{2n}\}$ is bounded away from zero, so we have $F(Sy_{2n} - x_{2n}) \to 0$; It also follows that $F(Sy_{2n} - z) \to 0$;Since T and S satisfies (1. 14) we have

$F(Tx_{2n} - Sy_{2n}) \leq aF(x_{2n} - y_{2n}) + bF(Tx_{2n} - x_{2n}) + cF(Sy_{2n} - y_{2n}) + eF(Tx_{2n} - y_{2n})$
$+ fF(Sy_{2n} - x_{2n})$ (1. 18)

Now $F(y_{2n} - x_{2n}) = F(\beta_{2n}Tx_{2n} + (1 - \beta_{2n})x_{2n} - x_{2n}) = F(\beta_{2n}Tx_{2n} - \beta_{2n}x_{2n})$
$= F(\beta_{2n}(Tx_{2n} - x_{2n})) \leq F(Tx_{2n} - x_{2n}) \leq F(Tx_{2n} - Sy_{2n} + Sy_{2n} - x_{2n})$
$\leq F(Tx_{2n} - Sy_{2n}) + F(Sy_{2n} - x_{2n})$ (1. 19)

$F(y_{2n} - Sy_{2n}) = F(\beta_{2n}Tx_{2n} + (1 - \beta_{2n})x_{2n} - Sy_{2n})$
$= F(\beta_{2n}Tx_{2n} + (1 - \beta_{2n})x_{2n} + \beta_{2n}Sy_{2n} - \beta_{2n}Sy_{2n} - Sy_{2n}) = F(\beta_{2n}(Tx_{2n} - Sy_{2n})$
$+ (1 - \beta_{2n})(x_{2n} - Sy_{2n})) \leq F(Tx_{2n} - Sy_{2n}) + F(x_{2n} - Sy_{2n})$ (1. 20)

$F(y_{2n} - Tx_{2n}) = F(\beta_{2n}Tx_{2n} + (1 - \beta_{2n})x_{2n} - Tx_{2n}) = F((1 - \beta_{2n})(Tx_{2n} - x_{2n}))$
$\leq F(Tx_{2n} - x_{2n}) \leq F(Tx_{2n} - Sy_{2n} + Sy_{2n} - x_{2n}) \leq F(Tx_{2n} - Sy_{2n}) + F(Sy_{2n} - x_{2n})$ (1. 21)

Using (1. 19), (1. 20) and (1. 21); (1. 18) can be written as
$F(Tx_{2n} - Sy_{2n}) \leq F(Tx_{2n} - Sy_{2n})$ contradiction
$\Rightarrow F(Tx_{2n} - Sy_{2n}) \to 0$ as $n \to \infty \Rightarrow Tx_{2n} - Sy_{2n} \to 0$ as $n \to \infty$
$\Rightarrow Tx_{2n} - z \to 0$ as $n \to \infty$ and $Sy_{2n} \to z$ if $n \to \infty$

If x_{2n}, z satisfies (1. 18), we have $F(Tx_{2n} - Sz) \leq aF(x_{2n} - z) + bF(Tx_{2n} - x_{2n}) + cF(Sz - z) + eF(Tx_{2n} - z) + fF(Sz - x_{2n})$ (1. 22) $\Rightarrow F(Tx_{2n} - Sz) \to 0$ as $n \to \infty$.

Finally $F(z - Sz) = F(z - Tx_{2n} + Tx_{2n} - Sz) \leq F(z - Tx_{2n}) + F(Tx_{2n} - Sz)$
$\Rightarrow F(z - Sz) \to 0$ as $n \to \infty$ So $z = Sz$, Similarly $z = Tz$.
Thus z is a common fixed point of T and S.

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