Selection Rejection Methodology For Three Dimensional Continuous Random Variables And Its Application To Three Dimensional Normal Distribution

Sachinandan Chanda

Department of Mathematics, Shillong Polytechnic, Mawlai, Shillong –793022

ABSTRACT

In this paper we have generalized the Selection-Rejection Methodology for two dimensional continuous random variables to three dimensional continuous random variables and applied it to the three dimensional normal distribution.

Keywords: random variable, iterations, target probability distribution and proposal probability distribution.

1. INTRODUCTION.


2. SELECTION-REJECTION METHODOLOGY FOR THREE DIMENSIONAL CONTINUOUS RANDOM VARIABLES.

Let $X,Y,Z$ be a two dimensional continuous random variable with probability distribution function $f(x,y,z) \forall x,y,z \in R$, where $R$=set of all real numbers. Let $g(x,y,z) \forall x,y,z \in R$ where $R$=set of all real numbers be another probability density
function such that \( \frac{f(x,y,z)}{g(x,y,z)} \leq k \quad \forall x, y, z \in R \), where \( k \geq 1 \) is a real number. By successively selecting different values of \( X, Y, Z \) we will try to make the ratio \( \frac{f(x,y,z)}{g(x,y,z)} \) as close to 1 as possible. The probability density function \( f(x,y,z) \) is called target distribution and the probability density function \( g(x,y,z) \) is called proposal distribution.

**The step by step procedure for the Selection-Rejection Methodology is as follows.**

**Step (1):** Let \( X, Y, Z \) be a three dimensional continuous random variable with probability distribution function \( f(x,y,z) \forall x, y, z \in R \), where \( R = \text{set of all real numbers} \).

**Step (2):** Let \( X', Y', Z' \) be another three dimensional continuous random variable (which is independent of \( X, Y, Z \)) with probability distribution function \( g(x,y,z) \forall x, y, z \in R \), where \( R = \text{set of all real numbers} \).

**Step (3):** Let \( \frac{f(X', Y', Z')}{g(X', Y', Z')} \leq k \quad \forall X', Y', Z' \in R \), where \( k \geq 1 \) a real number.

**Step (4):** Let \( 0 < R_1 < 1, 0 < R_2 < 1 \) and \( 0 < R_3 < 1 \) be three random numbers.

**Step (5):** Set \( X' \) in terms of \( R_1 \), set \( Y' \) in terms of \( R_2 \) and set \( Z' \) in terms of \( R_3 \) depending on the expression obtained for the ratio \( \frac{f(X', Y', Z')}{g(X', Y', Z')} \).

**Step (6):** If \( R_1 R_2 R_3 \leq \frac{f(X', Y', Z')}{g(X', Y', Z')} \), then set \( X, Y, Z = X', Y', Z' \) and select the continuous random variable \( X', Y', Z' \); otherwise reject the variable \( X', Y', Z' \) and repeat the process from step (1).

The probability that the continuous random variable \( X', Y', Z' \) is selected is \( \frac{1}{k} \).

The number of iterations required to select \( X', Y', Z' \) is \( k \).

It may be noted that \( 0 \leq \frac{f(X', Y', Z')}{g(X', Y', Z')} \leq 1 \)

**To prove that the probability for the selection of \( X', Y', Z' \) is \( \frac{1}{k} \).**

**Proof:** \( P \left[ \text{Select } X', Y', Z' \right] = P \left( R_1 R_2 R_3 \leq \frac{f(X', Y', Z')}{g(X', Y', Z')} \right) = \frac{f(X', Y', Z')}{g(X', Y', Z')} \ldots \)
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\[ P \ X',Y',Z' \ is \ selected = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{w,u,v} g_{w,u,v} \, dw \, du \, dv \]

\[ = \frac{1}{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{w,u,v} \, dw \, du \, dv = \frac{1}{k} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{w,u,v} \, dw \, du \, dv = 1 \right] \]

Hence the proof.

Since the probability of selection (i.e. success) is \( \frac{1}{k} \), the number of iterations needed will follow a geometric distribution with \( p = \frac{1}{k} \). So, on average it will take \( k \) iterations to generate a number.

3. APPLICATION TO THREE DIMENSIONAL NORMAL DISTRIBUTION.

Two dimensional normal distribution is given by

\[ f_{x,y,z} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2+y^2+z^2}{2}} \quad x \geq 0, \, y \geq 0, \, z \geq 0, \, x, \, y, \, z \in R \]  \( \text{(1)} \)

Here \( f_{x,y,z} \) is the target distribution.

Let \( g(x, y, z) = e^{-(x+y+z)} \), \( x \geq 0, \, y \geq 0, \, z \geq 0 \) be the proposal distribution.

Let \( h(x, y, z) = \frac{f(x, y, z)}{g(x, y, z)} = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2+y^2+z^2-2x-2y-2z}{2} \right) \)  \( \text{(2)} \)

With the help of differential calculus we can show that \( h(x, y, z) \) attains maximum at \( 1, 1, 1 \) and the maximum value of \( h(x, y, z) \) is

\[ \frac{3}{\sqrt{2\pi}} e^{\frac{3}{2}} \approx 1.7879 \quad \text{(approximately)} \]

Choosing \( k = \frac{3}{\sqrt{2\pi}} e^{\frac{3}{2}} \), we get

\[ \frac{f_{x,y,z}}{kg_{x,y,z}} = \exp \left( -\frac{x-1}{2} \right) \times \exp \left( -\frac{y-1}{2} \right) \times \exp \left( -\frac{z-1}{2} \right) \]  \( \text{(4)} \)
Selection-Rejection Methodology for the two dimensional distribution is as follows

**Step (1):** Let \( X, Y, Z \) be a two dimensional continuous random variable with probability distribution function \( f_{x,y,z} \) \( \forall x, y, z \in \mathbb{R} \), where \( \mathbb{R} \) = set of all real numbers.

**Step (2):** Let \( X', Y', Z' \) be a two dimensional continuous random variable with probability distribution function \( g_{x,y,z} \) \( \forall x, y, z \in \mathbb{R} \), where \( \mathbb{R} \) = set of all real numbers.

**Step (3):** Let \( 0 < R_1 < 1 \), \( 0 < R_2 < 1 \) and \( 0 < R_3 < 1 \) be three random numbers.

**Step (4):** Set \( X' = 1 + \sqrt{-2 \ln(R_1)} \), \( Y' = 1 + \sqrt{-2 \ln(R_2)} \), and \( Z' = 1 + \sqrt{-2 \ln(R_3)} \)

**Step (5):** If

\[
R_1 R_2 R_3 \leq \exp\left(-\frac{X'-1}{2}\right) \times \exp\left(-\frac{Y'-1}{2}\right) \times \exp\left(-\frac{Z'-1}{2}\right),
\]

then set \( X, Y, Z = X', Y', Z' \) and select \( X', Y', Z' \); otherwise reject \( X', Y', Z' \) and repeat the process from Step (1).

**Conclusion**
Selection-Rejection Methodology is valid for any dimension of random variable (continuous or discrete). In this method we approximate the target function to proposal function so that after a number of successive iterations the proposal function becomes almost equal to target function and proposal function is selected.

**References**


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