Extreme Edge-to-vertex Geodesic Graphs

S. Sujitha

Department of Mathematics, Holy Cross College(Autonomous),
Nagercoil- 629004, India.
email : sujivenkit@gmail.com

J. John

Department of Mathematics, Government College of Engineering,
Tirunelveli - 627007, India.
email : johnramesh1971@yahoo.co.in

A.Vijayan

Department of Mathematics, N.M Christian College,
Marthandam - 629165, India.
email : vijayan2020@yahoo.in

Abstract

For a connected graph \( G = (V, E) \), an edge-to-vertex geodetic basis \( S \) in a connected graph \( G \) is called an extreme edge-to-vertex geodetic basis if \( S \subseteq S_e \), where \( S_e \) denotes the set of all extreme edges of \( G \). A graph \( G \) is said to be an extreme edge-to-vertex geodesic graph if \( G \) contains at least one extreme edge-to-vertex geodetic basis. An edge-to-vertex geodetic basis \( S \) in a connected graph \( G \) is called a perfect extreme edge-to-vertex geodetic basis if \( S = S_e \). A graph \( G \) is said to be a perfect extreme edge-to-vertex geodesic graph if \( G \) contains a perfect extreme edge-to-vertex geodetic basis, that is, if \( G \) has an edge-to-vertex geodetic basis consisting of all the extreme edges of \( G \). Extreme edge-to-vertex geodesic graph \( G \) of size \( q \) with edge-to-vertex geodetic number \( q \) or \( q - 1 \) or \( q - 2 \) are characterized. It is shown that for each triple, \( d, k, q \) of integers with \( 2 \leq k \leq q - d + 2, \ d \geq 4 \), and \( q - d - k + 1 > 0 \), there exists a perfect extreme edge-to-vertex geodesic graph \( G \) of size \( q \) with \( \text{diam} \ G = d \) and \( g_v(G) = k \).

Keywords: distance, geodesic, edge-to-vertex geodetic basis, edge-to-vertex geodetic number.

AMS Subject Classification: 05C12.
1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1, 4]. A subset $M \subseteq E(G)$ is called a matching of $G$ if no pair of edges in $M$ are incident. The maximum size of such $M$ is called the matching number of $G$ and is denoted by $\beta'(G)$. An edge covering of $G$ is a subset $K \subseteq E(G)$ such that each vertex of $G$ is end of some edge in $K$. The number of edges in a minimum edge covering of $G$, denoted by $\beta'(G)$ is the edge covering number of $G$. For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in $G$. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices is the radius, $rad \ G$ and the maximum eccentricity is the diameter, $diam \ G$. A geodesic set of $G$ is a set $S$ of vertices such that every vertex of $G$ is contained in a geodesic joining some pair of vertices of $S$. The geodesic number $g(G)$ of $G$ is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a minimum geodetic set or simply a $g$-set of $G$. The geodetic number of a graph was introduced in [1] and further studied in [2,5]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. $N(v) = \{ u \in V(G) : uv \in E(G) \}$ is called the neighborhood of the vertex $v$ in $G$. A vertex $v$ is an extreme vertex of a graph $G$ if the subgraph induced by its neighbors is complete. The number of extreme vertices in $G$ is its extreme order $\text{ex}(G)$. A graph $G$ is said to be an extreme geodesic graph if $g(G) = \text{ex}(G)$, that is if $G$ has a unique minimum geodetic set consisting of the extreme vertices of $G$. The concept of extreme geodesic graphs is introduced in [3]. For subsets $A$ and $B$ of $V(G)$, the distance $d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B \}$. An $u - v$ path of length $d(A, B)$ is called an $A - B$ geodesic joining the sets $A$, $B$, where $u \in A$ and $v \in B$. A vertex $x$ is said to lie on an $A - B$ geodesic if $x$ is a vertex of an $A - B$ geodesic. For $A = \{ u, v \}$ and $B = \{ z, w \}$ with $uv$ and $zw$ edges, we write an $A - B$ geodesic as $uv - zw$ geodesic and $d(A, B)$ as $d(uv, zw)$. A set $S \subseteq E(G)$ is called an edge-to-vertex geodesic set if every vertex of $G$ is either incident with an edge of $S$ or lies on a geodesic joining a pair of edges of $S$. The edge-to-vertex geodesic number $g_{ev}(G)$ of $G$ is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an edge-to-vertex geodetic basis of $G$. The edge-to-vertex geodetic number of a graph was introduced in [9] and further studied in [6,8]. Since every edge covering of $G$ is an edge-to-vertex geodetic set of $G$, we have $g_{ev}(G) \leq \beta'(G)$. For an edge $e = uv \in E(G)$, $N(e) = N(u) \cup N(v)$. For a set $S \subseteq E(G)$, $N(S) = \{ N(e) : e \in S \}$. An edge $e$ of a graph $G$ is called an extreme edge of $G$ if one of its ends is an extreme vertex of $G$. Let $S_e$ denotes the set of all extreme edges of $G$, $E(e)$ denotes the number of extreme edges of $G$, and $c(G)$ denotes the length of the longest cycle in $G$. A double star is a tree with diameter three. A caterpillar is a tree or more, for which the removal of all end-vertices leaves a path.

**Example 1.1.** For the graph $G$ given in Figure 1.1 with $A = \{ v_4, v_5 \}$ and $B = \{ v_1, v_2, v_7 \}$, the paths $P : v_5, v_6, v_7$ and $Q : v_4, v_3, v_2$ are the only two $A - B$
geodesics so that \(d(A, B) = 2\).

**Example 1.2.** For the graph \(G\) given in Figure 1.2, the three \(v_1v_6 - v_3v_4\) geodesics are \(P : v_1, v_2, v_3; Q : v_1, v_2, v_4;\) and \(R : v_6, v_5, v_4\) with each of length 2 so that \(d(v_1v_6, v_3v_4) = 2\). Since the vertices \(v_2\) and \(v_5\) lie on the \(v_1v_6 - v_3v_4\) geodesics \(P\) and \(R\) respectively, \(S = \{v_1v_6, v_3v_4\}\) is an edge-to-vertex geodetic basis of \(G\) so that \(g_{ev}(G) = 2\).

The following theorems are used in sequel.

**Theorem 1.1.**[9] If \(v\) is an extreme vertex of a connected graph \(G\), then every edge-to-vertex geodetic set contains at least one extreme edge is incident with \(v\).

**Theorem 1.2.**[9] For any connected graph \(G\), \(g_{ev}(G) = q\) if and only if \(G\) is a star.

**Theorem 1.3.**[9] For any connected graph \(G\) with size \(q \geq 3\), \(g_{ev}(G) = q - 1\) if and only if \(G\) is either a double star or \(C_3\).

**Theorem 1.4.**[9] For a non-trivial tree \(T\) with \(k\) end-vertices, \(g_{ev}(T) = k\).

**Theorem 1.5.**[9] For any graph \(G\) of order \(p\), \(g_{ev}(G) \leq p - \alpha'(G)\).

## 2. Extreme Edge-to-Vertex Geodesic Graphs

**Definition 2.1.** An edge-to-vertex geodetic basis \(S\) in a connected graph \(G\) is called an extreme edge-to-vertex geodetic basis if \(S \subseteq S_e\). A graph \(G\) is said to be an extreme edge-to-vertex geodesic graph if \(G\) contains at least one extreme edge-to-vertex geodetic basis. An edge-to-vertex geodetic basis \(S\) in a connected graph \(G\) is called a perfect extreme edge-to-vertex geodetic basis if \(S = S_e\). A graph \(G\) is said to be a
perfect extreme edge-to-vertex geodesic graph if $G$ contains a perfect extreme edge-to-vertex geodesic basis, that is, if $G$ has an edge-to-vertex geodesic basis consisting of all the extreme edges of $G$.

**Example 2.2.** For the graph $G$ given in Figure 2.1(a), $S_e = \{v_1v_2, v_1v_6, v_3v_4, v_4v_5\}$. The set $S_1 = \{v_1v_2, v_4v_5\}$ is an edge-to-vertex geodesic basis of $G$. Since $S_1 \subseteq S_e$, $S_1$ is an extreme edge-to-vertex geodesic basis of $G$. Therefore, $G$ is an extreme edge-to-vertex geodesic graph. For the graph $G$ given in Figure 2.1(b), $S_e = \{v_1v_2, v_1v_7, v_4v_5\}$ is the unique extreme edge-to-vertex geodesic basis of $G$ so that $g_{ev}(G) = 3 = E(e)$. Therefore $G$ is a perfect extreme edge-to-vertex geodesic graph.

**Remark 2.3.** For an extreme edge-to-vertex geodesic graph $G$, there can be more than one extreme edge-to-vertex geodesic basis. For the graph $G$ given in Figure 2.1(a), $S_2 = \{v_1v_6, v_3v_4\}$ is an extreme edge-to-vertex geodesic basis.

![Figure 2.1](image)

For the complete graph $G = K_p(p \geq 3)$, every edge is an extreme edge. In [9], it is proved that, $g_{ev}(K_p)$ is either $p/2$ or $(p+1)/2$. So $K_p$ is an extreme edge-to-vertex geodesic graph. Since $g_{ev}(K_p) \neq E(e)$, $K_p$ is not a perfect extreme edge-to-vertex geodesic graph. A nontrivial tree $T$ has $k$ extreme edges, namely its end edges and so $E(e) = k$. Since $g_{ev}(G) = k$, it follows that $T$ is a perfect extreme edge-to-vertex geodesic graph. Obviously, a cycle $C_p(p \geq 4)$ has no extreme edges, a cycle is not an extreme edge-to-vertex geodesic graph. For any complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$, it is easily to see that no edge is an extreme edge and so $G$ is not an extreme edge-to-vertex geodesic graph.

**Theorem 2.4.** Let $G$ be an extreme edge-to-vertex geodesic graph of size $q \geq 2$ such that $d(e,f) = 0$ or $1$ for every $e,f \in E(G)$. Then $g_{ev}(G) = \beta'(G)$.

**Proof.** Let $S$ be an edge-to-vertex geodetic basis of $G$ and $v \in V(G)$. We claim that $v$ is incident with an edge of $S$. If not, then by Theorem 1.1, $v$ is not an extreme vertex of $G$. If $v \notin N(S)$, then $v$ lies on a xu- yw geodesic, where xu, yw $\in S$. Then it follows that $d(xu, yw) \geq 2$, which is a contradiction. Therefore $v \in N(S)$. Since $S$ is an edge-to-vertex geodetic basis of $G$ and since $d(e,f) = 0$ or $1$ for every $e,f \in E(G)$, the only
geodesics containing $v$ are $xvy$ and $xyvw$, where $xv, vy, xy, vw \in S$. This contradicts the
fact that $v$ is not incident with an edge of $S$. Therefore $v$ is incident with an edge of $S$. Which implies that $S$ is an edge covering of $G$ and so $\beta'(G) \leq g_{ev}(G)$. Hence $g_{ev}(G) = \beta'(G)$.

**Remark 2.5.** The converse of the Theorem 2.4 is not true. For the extreme edge-to-vertex geodesic graph $G$ given in Figure 2.2, $g_{ev}(G) = \beta'(G) = 6$ and $d(v_1v_2, v_8v_9) \geq 2$.

**Theorem 2.6.** Let $G$ be a connected graph of size $q \geq 2$. Then $G$ is a perfect extreme edge-to-vertex geodesic graph with edge-to-vertex geodetic number $q$ if and only if $G = K_1,q$.

**Proof.** This follows from Theorem 1.2.

**Theorem 2.7.** Let $G$ be a connected graph of size $q \geq 3$. Then $G$ is an extreme edge-to-vertex geodesic graph with edge-to-vertex geodetic number $q-1$ if and only if $G$ is either $C_3$ or a double star.

**Proof.** This follows from Theorem 1.3

**Theorem 2.8.** If $G$ is an extreme edge-to-vertex geodesic graph of size $q \geq 4$ and not a tree such that $g_{ev}(G) = q - 2$, then $G$ is unicyclic and $c(G) = 3$.

**Proof.** Let $G$ have more than one cycle. Then $q \geq p + 1$ and so $p - 1 \leq q - 2 = g_{ev}(G) \leq p - \alpha'(G)$, by Theorem 1.5. Hence $\alpha'(G) = 1$ and so $G$ must be either a star or the cycle $C_3$, a contradiction. Therefore $G$ is unicyclic. Then it follows from Theorem 1.5, $\alpha'(G) \leq 2$. Let $C_k$ be the unique cycle of $G$. We have $k \leq 5$ since otherwise $\alpha'(G) \geq \alpha'(C_k) \geq 3$. Therefore we have the following three cases:

**Case 1.** $k = 5$. Then $G$ cannot have any other vertices since otherwise $\alpha'(G) \geq 3$. Therefore $G = C_5$ which is not an extreme edge-to-vertex geodesic graph, which is a contradiction.

**Case 2.** $k = 4$. If $G = C_4$, then $G$ is not an extreme edge-to-vertex geodesic graph. So let $G \neq C_4$. Because $\alpha'(G) \leq 2$, only one of the vertices of $C_4$ has degree more than 2. Therefore $G$ is not an extreme edge-to-vertex geodesic graph, which is a contradiction. Therefore $c(G) = 3$.

**Theorem 2.9.** Let $G$ be a connected graph of size $q \geq 4$. Then $G$ is an extreme edge-to-vertex geodesic graph with edge-to-vertex geodetic number $q-2$ if and only if $G = K_1,q - 1 + e$ or caterpillar with diameter 4 or the graph $G$ given in Figure 2.3.
Proof. For a caterpillar of diameter 4, the result follows from Theorem 1.4. For $G = K_1q - 1 + e$, it follows from Theorem 1.1, that the set of all end edges of $G$ together with $e$ forms an edge-to-vertex geodesic basis so that $g_{ev}(G) = q - 2$. Further it is easily verified that $g_{ev}(G) = q - 2$ for the graph given in Figure 2.3.

Conversely let $G$ be an extreme edge-to-vertex geodesic graph such that $g_{ev}(G) = q - 2$. Then by Theorem 2.8, $G$ is either a tree or unicyclic. Let $G$ be a tree. Then it follows from Theorem 1.4 that $G$ has just two internal edges and hence $G$ is a caterpillar. Thus in this case the graph reduces to a caterpillar of diameter 4. Now, let $G$ be an unicyclic. By Theorem 2.8, $c(G) = 3$. Since $g_{ev}(C_3) = 2 = q - 1$, we have $G \neq C_3$. Let $V(C_3) = \{v_1, v_2, v_3\}$. We note that if $u \in V(G) - V(C_3)$, then $\deg u = 1$. Otherwise, there are $u_1, u_2 \in V(G) - V(C_3)$ such that $u_1$ is adjacent to both $u_2$ and $v_1$, say. Then it is easily seen that $E(G) - \{u_1v_1, v_1v_2, v_1v_3\}$ is an edge-to-vertex geodesic set, which implies that $g_{ev}(G) \leq q - 3$. Further at least one of $v_1$'s should be of degree 2. Otherwise $E(G) - E(C_3)$ is an edge-to-vertex geodesic set, which is impossible. Thus $G$ should be either $K_1q - 1 + e$ or a graph like Figure 2.3. The following theorem is proved in [9].

**Theorem A.** Let $G$ be a connected graph of size $q$ and diameter $d$, then $g_{ev}(G) \leq q - d + 2$.

If $G$ is a perfect extreme edge-to-vertex geodesic graph, then we have the following result.

**Theorem 2.10.** If $G$ is a perfect extreme edge-to-vertex geodesic graph of size $q$ and diameter $d$, then $E(e) \leq q - d + 2$.

Proof. Since $G$ is a perfect extreme edge-to-vertex geodesic graph, we have $g_{ev}(G) = E(e)$, now the result follows from Theorem A.

The following theorem characterize for trees.

**Theorem 2.11.** For any tree $T$, $g_{ev}(T) = q - d + 2 = E(e)$ if and only if $T$ is a caterpillar.

Proof. Let $P : v_0, v_1, ..., v_{d-1}, v_d = v$ be a diametral path of length $d$. Let $e_i = v_{i-1}v_i$ ($1 \leq i \leq d$) be the edges of the diametral path $P$. Let $k$ be the number of end edges of $T$ and $l$ be the number of internal edges of $T$ other than $e_i (2 \leq i \leq d - 1)$. Then $d - 2 + l + k = q$. By Theorem 1.4, $g_{ev}(T) = k = E(e)$ and so $g_{ev}(T) = q - d + 2 - l. Hence g_{ev}(T) = q - d + 2 = E(e)$ if and only if $l = 0$, if and only if all internal vertices of $T$ lie on the diametral path $P$, if and only if $T$ is a caterpillar.

In the following we give some realization results on perfect extreme edge-to-
vertex geodesic graphs.

**Theorem 2.12.** For every pair \( k, q \) of integers with \( 2 \leq k \leq q \), there exists a perfect extreme edge-to-vertex geodesic graph of size \( q \) with edge-to-vertex geodetic number \( q \).

**Proof.** For \( k = q \), the result follows from Theorem 2.6. Also, for each pair of integers with \( 2 \leq k < q \), there exists a tree of size \( q \) with \( k \) end edges. Hence the result follows from Theorem 1.4.

**Theorem 2.13.** For each triple, \( d, k, q \) of integers with \( 2 \leq k \leq q - d + 2 \), \( d \geq 4 \), and \( q - d - k + 1 > 0 \), there exists a perfect extreme edge-to-vertex geodesic graph \( G \) of size \( q \) with \( \text{diam} \ G = d \) and \( g_{ev}(G) = k \).

**Proof.** Let \( 2 \leq k = q - d + 2 \). Let \( G \) be the graph obtained from the path \( P \) of length \( d \) by adding \( q - d \) new vertices to \( P \) and joining them to any cut-vertex of \( P \). Then \( G \) is a tree of size \( q \) and \( \text{diam} \ G = d \). By Theorem 1.4, \( g_{ev}(G) = q - d + 2 = k \). Now, let \( 2 \leq k < q - d + 2 \).

**Case 1.** \( q - d - k + 1 \) is even. Let \( (q - d - k + 1) \geq 2 \). Let \( n = \frac{(q - d - k + 1)}{2} \). Then \( n \geq 1 \). Let \( P_d: u_0, u_1, \ldots, u_d \) be a path of length \( d \). Add new vertices \( v_1, v_2, \ldots, v_{k-2} \) and \( w_1, w_2, \ldots, w_n \) and join each \( v_i (1 \leq i \leq k - 2) \) with \( u_1 \) and also join each \( w_i (1 \leq i \leq n) \) with \( u_1 \) and \( u_3 \) in \( P_d \). Now, join \( w_1 \) with \( u_2 \) and we obtain the graph \( G \) in Figure 2.4(a). Then \( G \) has size \( q \) and diameter \( d \). By Theorem 1.1, all the end-edges \( u_1v_i (1 \leq i \leq k - 2), u_0u_1 \) and \( u_{d-1}u_d \) lie in every edge-to-vertex geodetic set of \( G \). Let \( S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_1u_0, u_{d-1}u_d\} \) be the set of all end-edges of \( G \). Then it is clear that \( S \) is an extreme edge-to-vertex geodetic set of \( G \) and so \( g_{ev}(G) = k \). Therefore \( G \) is a perfect extreme edge-to-vertex geodesic graph.

**Case 2.** \( q - d - k + 1 \) is odd. Let \( q - d - k + 1 \geq 5 \). Let \( m = (q - d - k) / 2 \). Then \( m \geq 2 \). Let \( P_d: u_0, u_1, \ldots, u_d \) be a path of length \( d \). Add new vertices \( v_1, v_2, \ldots, v_{k-2} \) and \( w_1, w_2, \ldots, w_m \) and join each \( v_i (1 \leq i \leq k - 2) \) with \( u_1 \) and also join each \( w_i (1 \leq i \leq m) \) with \( u_1 \) and \( u_3 \) in \( P_d \). Now join \( w_1 \) and \( w_2 \) with \( u_2 \) and we obtain the graph \( G \) in Figure 2.4(b). Then \( G \) has size \( q \) and diameter \( d \). Now, as in Case 1, \( S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\} \) is an extreme edge-to-vertex geodetic set of \( G \) so that \( g_{ev}(G) = k \). Therefore \( G \) is a perfect extreme edge-to-vertex geodesic graph.
Let \( q - d - k + 1 = 1 \). Let \( P_d : u_0, u_1, \ldots, u_d \) be a path of length \( d \). Add new vertices \( v_1, v_2, \ldots, v_{k-2} \) and \( w_1 \) and join each \( v_i (1 \leq i \leq k - 2) \) with \( u_1 \) and also join \( w_1 \) with \( u_1 \) and \( u_3 \) in \( P_d \), thereby obtaining the graph \( G \) in Figure 2.4(c). Then the graph is of size \( q \) and diameter \( d \). Now, as in Case 1, \( S = \{ u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_d; u_d \} \) is an extreme edge-to-vertex geodetic set of \( G \) so that \( g_{ev}(G) = k \). Therefore \( G \) is a perfect extreme edge-to-vertex geodesic graph.
Now, let $q - d - k + 1 = 3$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length $d$. Add new vertices $v_1, v_2, v_3, \ldots, v_{k-2}, w_1$ and $w_2$ and join each $v_i (1 \leq i \leq k - 2)$ with $u_1$ and also join $w_1$ and $w_2$ with $u_1$ and $u_3$ and obtain the graph $G$ in Figure 2.4(d). Then $G$ has size $q$ and diameter $d$. Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an extreme edge-to-vertex geodetic set of $G$ so that $g_{ev}(G) = k$. Therefore $G$ is a perfect extreme edge-to-vertex geodetic graph.

For every connected graph, $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$. Ostrand[7] showed that every two positive integers $a$ and $b$ with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand’s theorem can be extended to extreme to edge-to-vertex geodesic graphs.

**Theorem 2.14.** For positive integers $r$, $d$ and $l \geq 3$ with $r < d \leq 2r$, there exists a perfect extreme edge-to-vertex geodetic graph $G$ with $\text{rad } G = r$, $\text{diam } G = d$ and $g_{ev} = l = E(e)$.

**Proof.** When $r = 1$, let $G = K_{1, l}$. Then $d = 2$ and by Theorem 2.6, $g_{ev}(G) = l$ and $G$ is a perfect extreme edge-to-vertex geodetic graph. Now, let $r \geq 2$. Construct a graph $G$ with the desired properties as follows. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{2r}$ and $P_{d-r+1}$ by identifying $v_1$ in $C_{2r}$ and $u_0$ in $P_{d-r+1}$. Now, add $(l - 3)$ new vertices $w_1, w_2, \ldots, w_{l-3}$ to $H$ and join each vertex $w_i (1 \leq i \leq l - 3)$ to the vertex $u_{d-r-1}$ and join the vertices $v_r$ and $v_{r+2}$ and obtain the graph $G$ of Figure 2.5. Then $\text{rad } G = r$ and $\text{diam } G = d$. Let $S_e = \{v_r, v_{r+1}, v_{r+2}, u_{d-r-1}u_{d-r}, u_{d-r-1}w_1, u_{d-r-1}w_2, \ldots, u_{d-r-1}w_{l-3}\}$ be the set of $l$ extreme edges of $G$. Let $S_1 = S_e - \{v_r, v_{r+1}\}$ and $S_2 = S_e - \{v_{r+1}, v_{r+2}\}$. 
Then by Theorem 1.1, either $S_1$ or $S_2$ is a subset of every extreme edge-to-vertex geodetic set of $G$. It is clear that neither $S_1$ nor $S_2$ is an extreme edge-to-vertex geodetic set of $G$ and so $g_{ev} \geq l$. However, $S_e$ is an extreme edge-to-vertex geodetic set of $G$ so that that $g_{ev} = l$. Therefore $G$ is a perfect extreme edge-to-vertex geodesic graph.

\[ \begin{array}{c}
  v_1 \quad \cdots \quad v_{2r} \\
  v_{r+1} \\
  C_{2r} \\
  v_2 \\
  v_1 \equiv u_0 \\
  \cdots \\
  u_1 \quad u_2 \\
  u_{d-r} \quad u_{d-r-1} \\
  w_1 \quad \cdots \quad w_{l-3} \\
  G
\end{array} \]

Figure 2.5

References