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Some More Identities of Rogers-Ramanuan Type

Shaikh Fokor Uddin Ali Ahmed

Department of Mathematics,
F. A. Ahmed College, Garoimari, Kamrup, Assam (India)
E mail: fokoruddin86@gmail.com

ABSTRACT

In this paper, by using some definitions and results of Andrew V Sills [2], I have derived some identities of Rogers-Ramanujan Type, related to modulo 7, 9, and 11 by using the Jacobi's Triple Product Identity.

Key words: Rogers-Ramanujan Identity, Bailey pairs, Jacobi's Triple Product Identity.

1. Introduction:

For $|q| \leq 1$, the q-shifted factorial is denoted by

$(a;q)_0=1$, $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$ and

$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$.

It follows that

$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}$.

The multiple q-shifted factorial is

$(a_1, a_2, \ldots a_m)_n=(a_1;q)_n(a_2;q)_n \ldots \ldots (a_m;q)_n$

and

$(a_1, a_2, \ldots a_m)_\infty=(a_1;q)_\infty(a_2;q)_\infty \ldots \ldots (a_m;q)_\infty$

and throughout this paper we assume $|q| \leq 1$ to ensure convergence.

1.1 **The Rogers-Ramanujan Identity**:

The following two identities, namely for $|q| \leq 1$,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}\frac{1}{(q;q)_\infty}, \text{ where } n \not\equiv 0, \pm 2 \pmod 5 \quad (1)$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty}\frac{1}{(q;q)_\infty}, \text{ where } n \not\equiv 0, \pm 1 \pmod 5 \quad (2)$$

are the celebrated Rogers-Ramanujan Identity. These two identities, which have motivated extensive research over the past hundred years, were first discovered by L.

J. Rogers in 1894 and these were again rediscovered independently by S. Ramanujan and I. Schur.

In this regard, W. N. Bailey, G. N. Watson, L. J. Slater and many others has discovered several other identities of different modulo. Recently, the work of Andrew V. Sills has got a good recognition.

1.2. Bailey Pair: A pair of sequences $(\alpha_n(a;q), \beta_n(a;q))$ is called a Bailey pair if for n≥ 0, $\beta_n(a;q)=\sum_{r=0}^{n}\frac{\alpha_n(a;q)}{(q;q)_{n-r}(aq;q)_{n+r}}$

In [5] and [6], Bailey considered several Bailey pairs and proved several results including a fundamental result, now known as Bailey Lemma.

Bailey Lemma: If $(\alpha_r(a,q), \beta_j(a,q))$ form a Bailey pair, then

$$\frac{1}{(\frac{aq}{\rho_1};q)_n(\frac{aq}{\rho_2};q)_n}\sum_{j\geq 0}\frac{(\rho_1;q)_j(\rho_2;q)_j(\frac{aq}{\rho_1\rho_2};q)_{n-j}}{(q;q)_{n-j}}(\frac{aq}{\rho_1\rho_2})^j\beta_j(a,q)$$
$$=\sum_{r=0}^{n}\frac{(\rho_1;q)_r(\rho_2;q)_r}{(\frac{aq}{\rho_1};q)_r(\frac{aq}{\rho_2};q)_r(q;q)_{n-r}(aq;q)_{n+r}}(\frac{aq}{\rho_1\rho_2})^r\,\alpha_r(a,q). \quad (3)$$

1.3. **Jacobi's Triple Product Identity:(** see [7] 2.2.10 and 2.2.11)

$$(zq^{\frac{1}{2}}, z^{-1}q^{\frac{1}{2}};q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n.q^{\frac{n^2}{2}} \quad (4)$$

and its corollary

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in}$$
$$= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in}(1-q^{(2n+1)i})$$
$$=\prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}) \quad (5)$$

First we introduce the following two definitions and few results, which are due to Andrew V. Sills [2].

Andrew V. Sills ([2], p-9, eqn 18, 19, 20) observed the following definitions and two q-difference equations.

Definition 1: For k≥ 1, and 1≤i≤k

$$Q_{d,k,i}(a) = Q_{d,k,i}(a,q)$$
$$=\frac{1}{(aq;q)}\sum_{n\geq 0}\frac{(-1)^n a^{kn} q^{(dk+\frac{d}{2})n^2+(k-i+\frac{1}{2})dn}(1-a^i q^{(2n+1)di})(aq^d;q^d)_n}{(q^d;q^d)_n} \quad (6)$$

$$Q_{d,k,1}(a)=\frac{1}{(aq;q)_{d-1}}Q_{d,k,k}(aq^d) \quad (7)$$

And for 2≤i≤k,

$$Q_{d,k,i}(a)= Q_{d,k,i-1}(a) + \frac{a^{i-1}q^{(i-1)d}}{(aq;q)_{d-1}}Q_{d,k,k-i+1}(aq^d) \quad (8)$$

Moreover, the following result ([2], P-9, Lemma 3.2) also hold:

$$Q_{d,k,k}(a) = \frac{1}{(aq;q)_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{kn} q^{(dk+d/2)n^2-\frac{d}{2}n}(1-aq^{2dn})(a;q^d)_n}{(1-a)(q^d;q^d)_n} \quad (9)$$

Definition 2

$$F_{2,4,1}(a) = F_{2,4,1}(a;q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r}q^{n^2+2n+2r^2+2r}}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,2}(a) = F_{2,4,2}(a;q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r}q^{n^2+2n+2r^2+2r}(1+aq^{2r+2})}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,3}(a) = F_{2,4,3}(a;q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r}q^{n^2+2r^2+2r}}{(aq;q^2)_{n}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,4}(a) = F_{2,4,4}(a;q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r}q^{n^2+2r^2}}{(aq;q^2)_{n}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{3,3,1}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r a^n q^{n^2+3n+3r(r-1)/2}(aq^3;q^3)_{n-r}}{(aq;q)_{2n+2}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,3,2}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r a^{n-1} q^{n^2+3r(r-1)/2}(a;q^3)_{n-r}(1+aq^{3r}-q^{3r})}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,3,3}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r a^n q^{n^2+3r(r-1)/2}(a;q^3)_{n-r}}{(aq;q)_{2n-1}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,5,1}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r} q^{n^2+3n+3r^2+3r}(aq^3;q^3)_{n-r}}{(aq;q)_{2n+2}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,5,2}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r} q^{n^2+3n+3r^2+3r}(aq^3;q^3)_{n-r}(1+aq^{3r+3})}{(aq;q)_{2n+2}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,5,3}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r-1} q^{n^2+3r^2-3}(a;q^3)_{n-r}(q^{3r}+aq^{6r+3}-1)}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,5,4}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r} q^{n^2+3r^2+3r}(a;q^3)_{n-r}}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,5,5}(a) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{a^{n+r} q^{n^2+3r^2}(a;q^3)_{n-r}}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

Andrew V Sills [2] has correlated the Definition 1 and Definition 2 in the form of following three important results: (for proof see [2], theorem (3.12), (3.14) and (3.19))

For i=1, 2, 3, 4

$$F_{2,4,i}(a) = Q_{2,4,i}(a) \tag{10}$$

For i=1, 2 and 3

$$F_{3,3,i}(a) = Q_{3,3,i}(a) \tag{11}$$

And for i=1, 2, 3, 4 and 5

$$F_{3,5,i}(a) = Q_{3,5,i}(a) \tag{12}$$

Setting i=1, 2, 3, and 4 successively in (10), we get the following transformations:

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+2n+2r^2+2r}}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{9n^2+7n}(1-aq^{4n+2})(aq^2;q^2)_n}{(q^2;q^2)_n} \tag{13}$$

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+2n+2r^2+2r}(1+aq^{2r+2})}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{9n^2+5n}(1-a^2q^{8n+4})(aq^2;q^2)_n}{(q^2;q^2)_n} \tag{14}$$

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+2r^2+2r}}{(aq;q^2)_{n}(q;q)_{n-2r}(q^2;q^2)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{9n^2+3n}(1-a^3q^{12n+6})(aq^2;q^2)_n}{(q^2;q^2)_n} \tag{15}$$

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+2r^2}}{(aq;q^2)_n(q;q)_{n-2r}(q^2;q^2)_r}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{9n^2+n}(1-a^4q^{16n+8})(aq^2;q^2)_n}{(q^2;q^2)_n} \quad (16)$$

Now, setting a=1 and then replacing q by $q^{\frac{1}{2}}$ in all the above four transformations (13-16), we get the following identities:

$$\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+2n+2r^2+2r}{2}}}{(q^{\frac{1}{2}};q)_{n+1}(q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-2r}(q;q)_r}$$

$$=\frac{1}{(q,q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\frac{9n^2+7n}{2}}(1-q^{2n+1})$$

$$=\frac{1}{(q,q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{9n^2+7n}{2}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,1,8 \pmod 9 \quad (17)$$

(on using the Jacobi's Triple Product Identity)

$$\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+2n+2r^2+2r}{2}}(1+q^{\frac{4r+5}{2}})}{(q^{\frac{1}{2}};q)_{n+1}(q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-2r}(q;q)_r}$$

$$=\frac{1}{(q,q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{9n^2+5n}{2}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,2,7 \pmod 9 \quad (18)$$

(on using the Jacobi's Triple Product Identity)

$$\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+2r^2+2r}{2}}}{(q^{\frac{1}{2}};q)_n\,(q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-2r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,3,6 \pmod 9 \quad (19)$$

And,

$$\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+2r^2}{2}}}{(q^{\frac{1}{2}};q)_n\,(q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-2r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n\not\equiv 0,4,5 \pmod 9 \quad (20)$$

Moreover, setting d=2 and k =i= 4 in (9), we have

$$Q_{2,4,4}(a)=\frac{1}{(aq;q)_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{4n}q^{9n^2-n}(1-aq^{4n})(a;q^2)_n}{(1-a)(q^2;q^2)_n} \quad (21)$$

Now inserting (21) in the result (10) for i=4, we have

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+2r^2}}{(aq;q^2)_n(q;q)_{n-2r}(q^2;q^2)_r}$$

$$=\frac{1}{(aq,q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{9n^2-n}(1-aq^{4n})(a;q^2)_n}{(1-a)(q^2;q^2)_n} \quad (22)$$

Now setting a $=q^2$ and then replacing q by $q^{\frac{1}{2}}$ in (22), we get the following identity:

$$\frac{(q;q^{\frac{1}{2}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+2n+2r^2+2r}{2}}}{(q^{\frac{3}{2}};q)_n\,(q^{\frac{1}{2}};q^{\frac{1}{2}})_{n-2r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,1,8 \pmod 9 \qquad (23)$$

Again setting i=1, 2, and 3 successively in (11), we get the following transformations:

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r a^n q^{n^2+3n+3r(r-1)/2}}{(aq;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{3n} q^{\frac{21n^2+15n}{2}}(1-aq^{6n+3})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (24)$$

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r a^{n-1} q^{n^2+3r(r-3)/2}(a;q^3)_{n-r}(1+aq^{3r}-q^{3r})}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{3n} q^{\frac{21n^2+9n}{2}}(1-a^2q^{12n+6})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (25)$$

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r a^n q^{n^2+3r(r-1)/2}(a;q^3)_{n-r}}{(a;q)_{2n-1}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{3n} q^{\frac{21n^2+3n}{2}}(1-a^3q^{18n+9})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (26)$$

Now, setting a=1 and then replacing q by $q^{\frac{1}{3}}$ in all the above three transformations (24-26), we get the following identities:

$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r q^{\frac{n^2+3n+3r(r-1)/2}{3}}}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_{2n+2}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 \pmod 7 \qquad (27)$$

$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r q^{\frac{n^2+3r(r-3)/2}{3}}}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_{2n-1}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 2 \pmod 7 \qquad (28)$$

$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r q^{\frac{n^2+3r(r-1)/2}{3}}(q;q)_{n-r-1}}{(q^{\frac{1}{3}};q^{\frac{1}{3}})_{2n-2}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 3 \pmod 7 \qquad (29)$$

Also, using the result (9) for d= k = i= 3 in (11) for i=3, we get

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r a^n q^{n^2+3r(r-1)/2}(a;q^3)_{n-r}}{(a;q)_{2n-1}(q;q)_{n-3r}(q^3;q^3)_r}=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{3n} q^{\frac{21n^2-3n}{2}}(1-aq^{6n})(a;q^3)_n}{(1-a)(q^3;q^3)_n} \qquad (30)$$

Now setting a= q^3 and then replacing q by $q^{\frac{1}{3}}$ in (30), we get the following identity:

$$\frac{(q;q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{(-1)^r q^{\frac{n^2+3n+3r(r-3)/2}{3}}(q;q)_{n-r}}{(q;q^{\frac{1}{3}})_{2n-1}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 \pmod 7 \qquad (31)$$

Finally, setting i=1, 2, 3, 4and 5 successively in (12), we get the following transformations:

$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+3n+3r^2+3r}}{(aq;q)_{2n+2}(q;q)_{n-3r}(q^3;q^3)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n}q^{\frac{33n^2+27n}{2}}(1-aq^{6n+3})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (32)$$
$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+3n+3r^2+3r}\,(aq^3;q^3)_{n-r}(1+aq^{3r+3})}{(aq;q)_{2n+2}(q;q)_{n-3r}(q^3;q^3)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n}q^{\frac{33n^2+21n}{2}}(1-a^2q^{12n+6})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (33)$$
$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r-1}q^{n^2+3r^2-3}\,(a;q^3)_{n-r}\,(q^{3r}+aq^{6r+3}-1)}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n}q^{\frac{33n^2+15n}{2}}(1-a^3q^{18n+9})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (34)$$
$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+3r^2+3r}\,(a;q^3)_{n-r}}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n}q^{\frac{33n^2+9n}{2}}(1-a^4q^{24n+12})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (35)$$
$$\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{a^{n+r}q^{n^2+3r^2}\,(a;q^3)_{n-r}}{(a;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$
$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n}q^{\frac{33n^2+3n}{2}}(1-a^5q^{30n+15})(aq^3;q^3)_n}{(q^3;q^3)_n} \qquad (36)$$

Now, setting a=1 and then replacing q by $q^{\frac{1}{3}}$ in all the above five transformations (32-36), we get the following identities:

$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+3r^2+3n+3r}{3}}(q;q)_{n-r}}{(q^{1/3};q^{1/3})_{2n+2}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$
$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n \not\equiv 0, \pm1 \pmod{11} \qquad (37)$$
$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+3r^2+3n+3r}{3}}(q;q)_{n-r}(1+q^{r+1})}{(q^{1/3};q^{1/3})_{2n+2}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$
$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n \not\equiv 0, \pm2 \pmod{11} \qquad (38)$$
$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+3r^2-3}{3}}(q;q)_{n-r-1}(q^r+q^{2r+1}-1)}{(q^{1/3};q^{1/3})_{2n-1}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$
$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n \not\equiv 0, \pm3 \pmod{11} \qquad (39)$$
$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+3r^2+3r}{3}}(q;q)_{n-r-1}}{(q^{1/3};q^{1/3})_{2n-1}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$
$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n \not\equiv 0, \pm4 \pmod{11} \qquad (40)$$

And,

$$\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty}{(q,q)_\infty}\sum_{n=0}^{\infty}\sum_{r=o}^{\infty}\frac{q^{\frac{n^2+3r^2}{3}}(q;q)_{n-r-1}}{(q^{1/3};q^{1/3})_{2n-1}(q^{\frac{1}{3}};q^{\frac{1}{3}})_{n-3r}(q;q)_r}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n \not\equiv 0, \pm 5 \pmod{11} \tag{41}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n} q^{\frac{33n^2+27n}{2}}(1-aq^{6n+3})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{32}$$

$$=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^{5n} q^{\frac{33n^2+27n}{2}}(1-aq^{6n+3})(aq^3;q^3)_n}{(q^3;q^3)_n}$$

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