On Partial Quasi-bilateral Generating Functions Involving Gegenbauer Polynomials

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Abstract

In this paper, we have obtained a generalization of a known result on quasi-bilateral generating relation involving Gegenbauer polynomials from the existence of partial quasi-bilateral generating relation of the polynomial under consideration. Some particular cases of interest are also pointed out.

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1. Introduction

In [1], Mondal defined partial quasi-bilateral generating for two special functions by means of the relation:

\[ G(x, u, w) = \sum_{n=0}^{\infty} a_n P_{m+n}^{(\alpha)}(x) Q_r^{(m+n)}(u)w^n, \]

where \( a_n \), the coefficients are quite arbitrary and \( P_{m+n}^{(\alpha)}(x), Q_r^{(m+n)}(u) \) are two particular special functions of orders \( m+n, r \) and of parameters \( \alpha \) and \( m+n \) respectively. If \( Q_r^{(m+n)}(u) \equiv P_r^{(m+n)}(u) \), the generating relation is known as partial quasi-bilinear.
In this note, we would like to show that the existence of a partial quasi-bilinear generating function implies the existence of a more general generating function from the group theoretic view-point.

In [2], Samanta, Chandra and Bera have proved the following theorem on bilateral generating functions involving modified Gegenbauer polynomials, $C_{n+1}^\lambda (x)$ by group-theoretic method.

**Theorem 1** If there exists a unilateral generating relation of the form

$$ G(x, w) = \sum_{n=0}^{\infty} a_n C_{n+1}^\lambda (x) w^n $$  \hspace{1cm} (1.1)

then

$$ \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+w^2 x^2\}^{\lambda/2}} G \left( \frac{x}{\{1-w+w^2 x^2\}^{\lambda/2}}, \frac{w(1-w)}{\{1-w+w^2 x^2\}^{\lambda/2}} \right) = \sum_{n=0}^{\infty} w^n \sigma_n (x, v), \hspace{1cm} (1.2) $$

where

$$ \sigma_n (x, v) = \sum_{p=0}^{n} a_p \left( \frac{p+1}{2} \right)_{n-p} \left( \frac{p+2}{2} \right)_{n-p} C_{n+1}^\lambda (x) v^p. $$

The importance of the above theorem lies in the fact that whenever one knows a unilateral generating relation of the form (1.1) then the corresponding bilateral generating relation can at once be written down from (1.2). So one can get a large number of bilateral generating relations by attributing different suitable values to $a_n$ in (1.1).

Subsequently, In [3], Samanta and Chongdar obtained an extension of the above theorem in the following form:

**Theorem 2** If there exists a unilateral generating relation of the form

$$ G(x, w) = \sum_{n=0}^{\infty} a_n C_{n+r}^\lambda (x) w^n $$  \hspace{1cm} (1.3)

then

$$ \frac{(1-w)^{\lambda-\frac{1}{2}}}{\{1-w+w^2 x^2\}^{\lambda/2}} G \left( \frac{x}{\{1-w+w^2 x^2\}^{\lambda/2}}, \frac{w(1-w)}{\{1-w+w^2 x^2\}^{\lambda/2}} \right) = \sum_{n=0}^{\infty} w^n \sigma_n (x, v), \hspace{1cm} (1.4) $$

Where

$$ \sigma_n (x, v) = \sum_{p=0}^{n} a_p \left( \frac{p+r+1}{2} \right)_{n-p} \left( \frac{p+r+2}{2} \right)_{n-p} C_{n+r}^\lambda (x) v^p. $$
In [4], authors have obtained a nice extension of the Theorem 1 from the existence of quasi-bilinear generating relation.

**Theorem 3** If there exists a quasi-bilinear generating relation of the following form

\[ G(x, u, w) = \sum_{n=0}^{\infty} a_n c_{n+1}(x) c_m(u) w^n \]

then

\[ (1 - 2w)^{\lambda - \frac{m}{2}} \left( 1 - 2w(1 - x^2) \right)^{-\lambda} \times G \left( \frac{x}{\left( 1 - 2w(1 - x^2) \right)^{\frac{1}{2}}}, \frac{u}{(1 - 2w)^{\frac{1}{2}}}, \frac{wt}{\left( 1 - 2w(1 - x^2) \right)^{\frac{3}{2}}} \right) \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{(n+1)}{p} \frac{(n+p+q)}{p} \frac{(n+q)}{p} c_{n+2p}^{\lambda + n - p}(x) c_{m+q}^{n+r+q}(u) t^n. \]

The object of the present paper is to further generalize the above theorem from the concept of partial quasi-bilateral (or partial quasi-bilinear) generating functions. In fact, we have obtained the following theorem as the main result of our investigation.

**Theorem 4** If there exists a partial quasi-bilinear generating relation of the following form

\[ G(x, u, w) = \sum_{n=0}^{\infty} a_n c_{n+r}(x) c_m(u) w^n \]

Then

\[ (1 - 2w)^{\lambda - \frac{m}{2} - \frac{1}{2}} \left( 1 - 2w(1 - x^2) \right)^{-\lambda} \times G \left( \frac{x}{\left( 1 - 2w(1 - x^2) \right)^{\frac{1}{2}}}, \frac{u}{(1 - 2w)^{\frac{1}{2}}}, \frac{wt}{\left( 1 - 2w(1 - x^2) \right)^{\frac{3}{2}}} \right) \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} 2^{p+q} \frac{(n+r+1)}{p} \frac{(n+r+q)}{q} \frac{(n+r+p)}{p} c_{n+r+2p}^{\lambda + n - p}(x) c_{m+q}^{n+r+q}(u) t^n. \]

**2. Proof of the theorem**

At first we consider the generating relation of the form:

\[ G(x, u, w) = \sum_{n=0}^{\infty} a_n c_{n+r}(x) c_m(u) w^n. \quad (2.1) \]

For the Gegenbauer polynomials, we consider the following operators[3,5]:
Now, replacing $\lambda$ with $(2.7)$, with the help of $(2.4)$ and $(2.5)$, becomes

\begin{align*}
R_1 &= x(1 - x^2) \frac{y^2}{z^3} \frac{\partial}{\partial x} + (1 - 3x^2) \frac{y^3}{z^3} \frac{\partial}{\partial y} - \frac{2x^2y^2}{z^2} \frac{\partial}{\partial z} + (1 + r - r x^2) \frac{y^2}{z^3}, \\
R_2 &= uv \frac{\partial}{\partial u} + 2v^2 \frac{\partial}{\partial v} + (m + 2r) v
\end{align*}

such that

\begin{align*}
R_1\left(C_n^{\lambda+n}(x)y^n z^\lambda\right) &= \frac{(n + r + 1)(n + r + 2)}{2(1 - \lambda - n)} C_n^{\lambda+n+1}(x)y^{n+2} z^{\lambda-3}, \quad (2.2) \\
R_2\left(C_m^n(u)v^n\right) &= 2(n + r) C_m^{n+r+1}(u)v^{n+1} \quad (2.3)
\end{align*}

And

\begin{align*}
e^{wR_1} f(x, y, z) &= \left(1 - 2w \frac{y^2}{z^3}\right)^{-\frac{1}{2}} \left(1 - 2w(1 - x^2) \frac{y^2}{z^3}\right)^{-r} \\
&\times \left(\frac{x}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}\right)^{\frac{1}{2}} \left(\frac{y}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}\right)^{\frac{1}{2}} \left(\frac{z}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}\right)^{\frac{1}{2}}, \quad (2.4)
\end{align*}

\begin{align*}
e^{wR_2} f(u, v) &= (1 - 2wv)^{-\frac{m}{2} - r} f\left(\frac{u}{1 - 2wv}, \frac{v}{1 - 2wv}\right) \quad (2.5)
\end{align*}

Replacing $w$ by $wvyt$ and multiplying both sides of (2.1) by $z^\lambda$, we get

\begin{align*}
z^\lambda G(x, u, wvyt) &= \sum_{n=0}^{\infty} a_n \left(C_n^{\lambda+n}(x)y^n z^\lambda\right)(C_m^{n+r}(u)v^n) (wt)^n. \quad (2.6)
\end{align*}

Now operating $e^{wR_1}$ $e^{wR_2}$ on both sides of (2.6), we get

\begin{align*}
e^{wR_1} e^{wR_2} [z^\lambda G(x, u, wvyt)] = \\
e^{wR_1} e^{wR_2} \left\{\sum_{n=0}^{\infty} a_n \left(C_n^{\lambda+n}(x)y^n z^\lambda\right)(C_m^{n+r}(u)v^n) (wt)^n\right\} \quad (2.7)
\end{align*}

The left member of (2.7), with the help of (2.4) and (2.5), becomes

\begin{align*}
&\left(1 - 2w \frac{y^2}{z^3}\right)^{-\frac{1}{2}} \left(1 - 2w(1 - x^2) \frac{y^2}{z^3}\right)^{-r} \left(1 - 2wv\right)^{-\frac{m}{2} - r} z^\lambda \\
&\times G\left(\frac{x}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}, \frac{u}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}, \frac{wvyt(1 - 2w \frac{y^2}{z^3})}{1 - 2w(1 - x^2) \frac{y^2}{z^3}}\right) \quad (2.8)
\end{align*}
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The right member of (2.7), with the help of (2.2) and (2.3), becomes

\[
\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n w^{p+q} \frac{n+r+1}{2^p} \frac{n+r+2}{2^p} \frac{n+r}{q} \frac{n+r+q}{(1-\lambda-n)_p} \times C_{n+r+2p}(x) C_m^{n+r+q}(u) y^{n+2p} z^{3p} v^{n+q} (wt)^n. \tag{2.9}
\]

Now equating both members, and then substituting \( y = z = v = 1 \), we get

\[
(1 - 2w)^{\lambda - \frac{m}{2} - \frac{1}{2} - r}\{1 - 2w(1 - x^2)\}^{-\frac{r}{2} - \lambda} \times G \left( \frac{x}{1 - 2w(1 - x^2)}, \frac{u}{1 - 2w}, \frac{wt}{1 - 2w(1 - x^2)} \right) = \\
\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n w^{n+p+q} \frac{n+r+1}{2^p} \frac{n+r+2}{2^p} \frac{n+r}{q} \frac{n+r+q}{(1-\lambda-n)_p} C_{n+r+2p}(x) C_m^{n+r+q}(u) t^n. \tag{2.10}
\]

This completes the proof of Theorem 4.

**Corollary 1:** Putting \( r = 0 \) in (10), we get

\[
(1 - 2w)^{\lambda - \frac{m}{2} - \frac{1}{2}}\{1 - 2w(1 - x^2)\}^{-\lambda} \times G \left( \frac{x}{1 - 2w(1 - x^2)}, \frac{u}{1 - 2w}, \frac{wt}{1 - 2w(1 - x^2)} \right) = \\
\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n w^{n+p+q} \frac{n+1}{2^p} \frac{n+2}{2^p} \frac{n}{q} \frac{n}{(1-\lambda-n)_p} C_{n+2p}(x) C_m^{n+q}(u) t^n,
\]

which is Theorem 3. Thus Theorem 4 is an extension of Theorem 3.

**Corollary 2:** If we put \( m = 0 \), we notice that \( G(x, u, w) \) becomes \( G(x, w) \) since \( C_0^{n+r+q}(u) = 1 \). Hence from (2.10), we get

\[
(1 - 2w)^{\lambda - \frac{1}{2} - r}\{1 - 2w(1 - x^2)\}^{-\frac{r}{2} - \lambda} \times G \left( \frac{x}{1 - 2w(1 - x^2)}, \frac{wt}{1 - 2w(1 - x^2)} \right)
\]
\[ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!} 2p \frac{(n+r+1)}{2} p \frac{(n+r+2)}{2} p C_{n+r+2p}(x) t^n \left( \sum_{q=0}^{\infty} \frac{(2w)^q (n+r)_q}{q!} q \right) \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{(2w)^{n+p}}{p!} p \frac{(n+r+1)}{2} p \frac{(n+r+2)}{2} p C_{n+r+2p}(x) \left( \frac{t}{2(1-2w)} \right)^n (1 - 2w)^{-r} \]

Replacing \( \frac{t}{2(1-2w)} \) by \( v \) and then \( 2w \) by \( w \) on both sides, we get

\[ (1-w)^{\lambda - \frac{1}{2}} \{ 1 - w(1-x^2) \}^{r-1} (1-w)^{-\lambda} G \left( \frac{x}{1-w(1-x^2)} \right), \frac{wv(1-w)}{1-w(1-x^2)} \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{(w)^{n+p}}{p!} p \frac{(n+r+1)}{2} p \frac{(n+r+2)}{2} p C_{n+r+2p}(x) v^n \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_n \frac{w^n}{p!} p \frac{(n-p+r+1)}{2} p \frac{(n-p+r+2)}{2} p C_{n+r+p}(x) v^n. \]

Therefore we have

\[ (1-w)^{\lambda - \frac{1}{2}} \{ 1 - w(1-x^2) \}^{r-1} (1-w)^{-\lambda} G \left( \frac{x}{1-w(1-x^2)} \right), wv(1-w) \]

\[ = \sum_{n=0}^{\infty} w^n \sigma_n(x,v) \]

Where

\[ \sigma_n(x,v) = \sum_{p=0}^{n} a_p \frac{(p+r+1)}{n-p} \frac{(p+r+2)}{2} p C_{n+r+p}(x) v^n. \]

**Corollary 3:** If we put \( r = 0 \) in the above result, we get the Theorem 1.

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References


