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Lotka Volterra Model with Time Delay

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Abstract

We model the Lotka-Volterra equations with a time delay. For this model we derive the conditions for the existence of steady state conditions, asymptotic stability.

1. Introduction

Time delay is inherent in the predator-prey systems. Both the predator and prey require their own times to grow from birth to maturity and from maturity to death. If the time spans for predator and prey are greatly different it will lead to instability in the predator–prey system. Using a time delay of a predator–prey system we derive the conditions for temporal stability of a predator–prey system following the techniques given in [1]. Predator prey models with time delay (or lag) have been studied by a number of authors [2]–[8]. Time delay can induces instability, oscillations and bifurcation and in some cases switching of stabilities. However most authors have used a single time delay to characterize the predator–prey system. Here we have associated unique delays with both predator and prey.

2. Time Delay Equations for Predator-Prey

The lotka Volterra equations with time delay are given by

$$\frac{d\psi_1(t)}{dt} = \psi_1(t)\left[\alpha - \beta\psi_2(t-t_1)\right] \tag{1}$$

$$\frac{d\psi_2(t)}{dt} = -\psi_2(t)\left[\gamma - \delta\psi_1(t-t_2)\right] \tag{2}$$

Here t_1, t_2 are delay times for predator and prey respectively.

On substituting

$$\psi_1(t) = e^{-\lambda t_1} A_1 \text{ and } \psi_2(t) = e^{-\lambda t_2} A_2 \tag{3}$$

we obtain

$$(\lambda+\alpha)\psi_1(t) - \beta A_1 \psi_2(t) e^{-\lambda(t+t_1)} = 0 \tag{4}$$

$$\delta A_2 e^{-\lambda(t+t_2)} \psi_1(t) + (\lambda-\gamma)\psi_2(t) = 0 \tag{5}$$

Here A_1, A_2 correspond to the predator, prey population at time t =0.

For solutions of $\psi_1(t)$ and $\psi_2(t)$ to exist, we must have, from equations (4) and (5), that

$$\begin{vmatrix} \lambda+\alpha & -\beta A_1 e^{-\lambda(t+t_1)} \\ \delta A_2 e^{-\lambda(t+t_2)} & \lambda-\gamma \end{vmatrix} = 0 \tag{6}$$

From equation (6), we get

$$\lambda^2 + a\lambda - b + \delta' e^{-\lambda(2t+t_1+t_2)} = 0 \tag{7}$$

$$\text{Where } \left. \begin{aligned} a &= |\alpha-\gamma| \\ b &= -\gamma\alpha \\ \delta' &= \delta\,\beta A_1 A_2 \end{aligned} \right\} \tag{8}$$

Note that we have defined $a = |\alpha-\gamma|$ as the time delay (which is defined in terms of a) has to be positive.

In absence of delay $\delta' \cos y(t_1+t_2) = \frac{1}{2}\left[\left(2b-a^2\right)+\sqrt{\Delta}\right] - b$ then the equation (7) becomes

$$P(\lambda,t) = \lambda^2 + a\lambda + b + \delta' e^{-\lambda(t_1+t_2)} = 0 \tag{9}$$

Suppose that $\lambda = iy$ (y > 0) is a purely imaginary root of equation (9).It suffices to seek solution $y > 0$.Putting $\lambda = iy$ in (9) and separating real parts and imaginary parts, we get

$$y^2 - b - \delta' \cos(t_1+t_2) = 0 \tag{10}$$

$$ay - \delta' Sin(t_1+t_2) = 0 \;\; (11)$$

Eliminating $t_1 + t_2$ from Equations (11) & (12),we get

$$U(y) = y^4 + \left(a^2 - 2b\right)y^2 + \left(b^2 - \delta'^2\right) = 0 \tag{12}$$

Let Δ be the determinant of equation (13),we get

$$\Delta = \left(a^2 - 2b\right)^2 - 4\left(b^2 - \delta'^2\right)$$

If $\left(a^2 - 2b\right)^2 < 4\left(b^2 - \delta'^2\right)$,then there is no positive solution of y^2.

If $\left(a^2 - 2b\right)^2 - 4\left(b^2 - \delta'^2\right) = \Delta > 0$ or if $\Delta = 0$ and $a^2 - 2b < 0$,then there may be a positive solution of y^2 given by

$$y^2 = \frac{\left(2b - a^2\right) \pm \sqrt{\Delta}}{2} \tag{13}$$

If $b^2 < \delta'^2$, then there is one positive solution of y^2 given by

$$y = \frac{1}{\sqrt{2}}\left[\left(2b - a^2\right) + \sqrt{\Delta}\right]^{\frac{1}{2}} \text{ (taking only positive sign)} \tag{14}$$

Thus $y > 0$

From equation (11), we get $\sin y\left(t_1 + t_2\right) = \dfrac{ay}{\delta'} > 0$ (15)

From equation (12), we get

$$\delta' \cos y\left(t_1 + t_2\right) = \frac{1}{2}\left[-a^2 + \sqrt{\Delta}\right] \tag{16}$$

Equation (16) determines the sign of $\cos y\left(t_1 + t_2\right)$ and the quadrant in which $\cos y\left(t_1 + t_2\right)$ must lie.

From equation (15), we get

$$\tau_n = \frac{1}{y}\left[\sin^{-1}\left(\frac{ay}{\delta'}\right) + 2n\pi\right]; n = 0, \pm 1, \pm 2, \pm 3, ... \tag{17}$$

Where $\tau = t_1 + t_2$ (18)

Where y is given by equation (14) and where $\sin^{-1}\left(\frac{ay}{\delta'}\right)$ is chosen in the first or second quadrant according to the sign of $\cos y(t_1+t_2)$

3. Stability Analysis

To analyze the change of behavior of the stability of the steady state E_0 with respect to t,we examine the sign of $\frac{d\lambda}{d\tau}$,where $\lambda = iy$.

If $\frac{d\lambda}{d\tau} > 0$,then the system is unstable for some values of τ.

If $\frac{d\lambda}{d\tau} < 0$,then is the system stable for some values of τ.

Differentiating equation (10) with respect to t,we get

$$\frac{d\lambda}{d\tau} = \frac{\delta'\tau e^{-\lambda\tau}}{2\lambda + a - \delta'\tau e^{-\lambda\tau}}$$

$$\text{Hence } \left(\frac{d\lambda}{d\tau}\right)_{at\,\lambda=iy} = \frac{4\delta' iy(\cos y\tau - i\sin y\tau)}{2iy + a - \delta'\tau(\cos y\tau - i\sin y\tau)} \tag{19}$$

Using (15) & (16) in (19) and simplifying, we get

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)_{at\ \lambda=iy}\right] = -\frac{\left[\left(a^2-\sqrt{\Delta}\right)\left(2a + a^2 - \sqrt{\Delta} + 2ay^2(4+a)\right)\right]}{\left(2a + a^2 - \sqrt{\Delta}\right)^2 + (4+a)^2 y^2} \tag{20}$$

Since $y^2 > 0$ and $\sqrt{\Delta} > 0$, $\mathrm{Re}\left[\left(\frac{d\lambda}{dt}\right)_{at\ \lambda=iy}\right] < 0$ at every root.

Therefore every time a root crosses the imaginary axis with increasing t it crosses from left to right, stability of the zero solution is lost at $\tau = \tau_0$,where

$$\tau_0 = \frac{1}{y}\left[\sin^{-1}\left(\frac{ay}{\delta'}\right)\right] \tag{21}$$

and stability persists for all $\tau \geq \tau_0$.

4. Conclusion

We have derived the stability criteria for a delayed predator–prey system. Note that τ is the composite delay as given by (19). Criteria for stability is given in (22). For the case $A_1 \approx 0$ or $A_2 \approx 0$ one finds that $\tau_0 = \frac{\sqrt{2}|\alpha-\gamma|}{\delta\beta A_1 A_2}$. In the opposite limit where A_1, A_2 are very large one obtains $\tau_0 = \frac{|\alpha-\gamma|}{\sqrt{2}\delta\beta\sqrt{A_1 A_2}}$ Note that in both limits if $\alpha = \gamma, \tau_0 = 0$. Here α is the rate of growth of predator and γ is the rate of death of prey. In other words in this limit there is no possibility of any predator –prey oscillations.

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