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A Study on the Properties of Rational Triangles

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Abstract

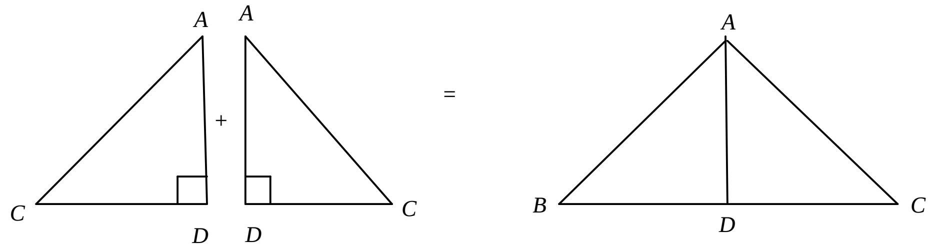
In this manuscript an attempt has been made to collaborate geometry and rational number system. In support of this some properties of rational triangles have been discussed and some theorems have been established following the geometrical concept of Brahme Gupta (b 598 AD), Bachet and Vieta etc.

Keywords: Opposite Parity, Primitive triangle, Rational triangle, Juxtaposition and Oblique triangle.

1. Introduction

Let us define some key terms to make the manuscript more understandable. The definitions are as follows:

- **Rational triangle: -** A triangle with rational sides and rational area is called a rational triangle.
- **Juxtaposition: -** The process of combination of two right triangles with a common leg is called Juxtaposition. The combined triangle is called oblique triangle.
- **Example: -** Let ADB &ADC be two right triangles with common leg AD and Right <ed at D, then the combined triangle ABC is the oblique triangle.



Juxtaposition *Oblique triangle*

- **Oblique Triangle:** After juxtaposition of two right triangles, the combined triangle is called oblique triangle.
- **Opposite parity:** Two relatively prime numbers are said to be of opposite parity if one of them is odd and the other is even.

  Ex.: 2 & 3 are of opposite parity, 3 & 4 are of opposite parity.

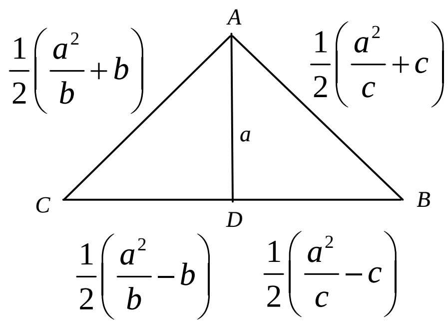
- **Primitive triangle:** A right triangle with relatively prime integral sides is called a primitive triangle.

  Ex.: Pythagorean triplet (3, 4, 5) represents a primitive right triangle as (3, 4, 5) = 1.

Section: - A

Early works about rational triangles

Brahme Gupta (b.598-AD) chose any three positive rational numbers a, b, c and established that $\frac{1}{2}\left(\frac{a^2}{b}+b\right)$, $\frac{1}{2}\left(\frac{a^2}{c}+c\right)$, $\frac{1}{2}\left(\frac{a^2}{b}-b\right)$ and $\frac{1}{2}\left(\frac{a^2}{c}-c\right)$ are the side of an oblique triangle whose altitude and area are rational, where the oblique triangle is formed by juxtaposition of two right triangles ADC &ABD with the common height AD = a.



C. G. Bachet [2] introduced two methods to solve rational triangles as follows:

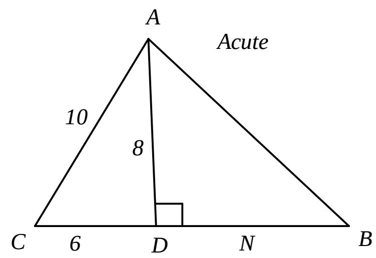
Method 1: In the first method Bachet considered a right triangle ADC with the sides 10, 8, 6 and BD = N such that $N^2 + 8^2 = AB^2$, where AB is rational.
From the properties of triangles, < A is acute or obtuse according as

$$\frac{CD}{AD} < \frac{AD}{BD} \;\; or \;\; \frac{CD}{AD} > \frac{AD}{BD}$$

$$i.e.\ 6N < 8^2 \;\; or \;\; 6N > 8^2$$

$$\therefore \;\; \angle A \; is \; acute \; if \;\; N < 32/3 \;\&\; \angle A \; is \; obtuse \, if \;\; N > 32/3.$$

Case 1: When $N < 32/3$



Let us choose a positive rational number x such that

$$N^2 + 8^2 = (8 - xN)^2 \tag{1}$$

then, $\dfrac{16x}{x^2-1} = N < 32/3$

$$\Rightarrow (2x+1)(x-2) > 0$$

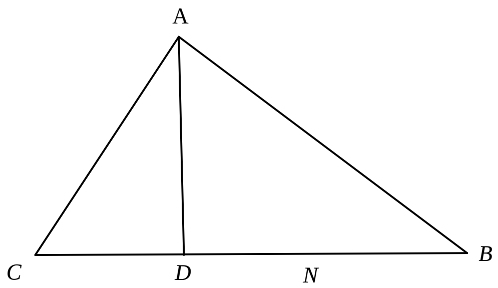
$$\Rightarrow x > 2 \qquad (as \;\; x > 0)$$

Ex.: Let x = 5/2, then from (1) $N = 160/21$

Thus the sides of the triangles ABC with acute $\angle$A and altitude 8 are 10, 6+N, $\sqrt{8^2 + N^2}$

i.e., 10, 286/21 and 232/21 are the rational sides.

Case 2: When $N > 32/3$



Similar to case (1)

$$N^2+8^2=(8-xN)^2$$

$$\frac{16x}{x^2-1}=N>32/3$$

$$\Rightarrow (2x+1)(x-2)<0$$

$$\Rightarrow x<2 \qquad (as\, x>0)$$

Ex: Let $x=3/2$, then from (1)

$$N^2+8^2=(8-3N^2/2)$$

Thus the sides of the triangle ABC with obtuse $\angle$A and altitude AD = 8 are $10, 6+\frac{96}{5}, \sqrt{8^2+\left(\frac{96}{5}\right)^2}$ *i.e.* $10, 126/5, 104/5$ are rational sides.

Method 2:

In the second method Bachet used the method of juxtaposition of two rational right triangles with common height AD. He considered AD = 12 and other two numbers x and y such that

$$x^2+12^2 = any\ square\ and\ y^2+12^2 = any\ square$$

$$Thus\ x=35 \Rightarrow 35^2+12^2=37^2$$

$$\&\quad y=16,\ 16^2+12^2=20^2$$

Using $(m^2-n^2)^2 + (2mn)^2 = (m^2+n^2)^2$

Hence one can find the rational triangle by juxtaposing as 37, 20, 35 + 16 = 51 with altitude 12.

Similarly by juxtaposing the triangles

$$\left[\,35^2+12^2=37^2 \,\&\, 9^2+12^2=15^2\right] or \left[35^2+12^2=37^2 \,\&\, 5^2+12^2=13^2\right],$$

We get the rational triangles (37, 15, 35+9) or (37, 13, 35+5).

F. Vieta's Method:

F. Vieta [3] obtained rational triangle by juxtaposing two right triangles as follows:

He considered a rational right triangle with right legs b and d and hypotenuse z and a second right triangle having altitude a = 2b (f + d), then the hypotenuse and base of the second triangle are $\left(f+d\right)^2+b^2$ & $\left(f+d\right)^2-b^2$ respectively.

Now multiplying the sides of the first triangle by a and the sides of the second triangle by d then the two right triangles become of common altitude ad.

Thus by Juxtaposition of two resulting right triangles with common attitude we obtained the rational oblique triangle with sides az., $d[(f+d)^2+b^2], d[(f+d)^2-b^2]+ab$.

For Ex, taking b = 3, d = 4, z = 5, f = 6, so that a = 60and ignoring the proportional factor 4, we get the rational triangle of sides 75, 109, 136 with altitude 60.

Section B (present works about rational triangles)

In this section we have established some new theorems applying the concept of juxtaposition of two right triangles and solved some oblique triangles.

Theorem (1)

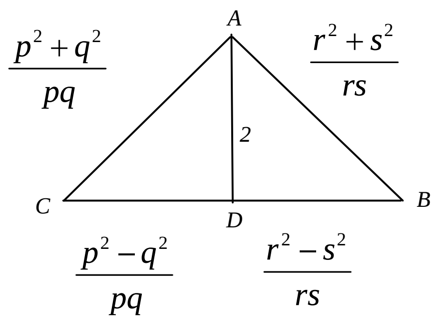
Statement:

In any triangle ABC with rational sides a, b, c and rational area.

$$a:b:c=\frac{[(ps\pm qr)(pr\pm qs)]}{pqrs}:\frac{p^2+q^2}{pq}:\frac{r^2+s^2}{rs},$$

where p, q, r, and s are some rational numbers and every pair of sides are in the ratio of two numbers of the form $\frac{u^2+v^2}{uv}$.

Proof:



Considering two right triangles ADC and ADB with sides

$$2,\frac{p^2+q^2}{pq}\frac{p^2-q^2}{pq}\ and\ 2,\frac{r^2+s^2}{rs},\ \frac{r^2-s^2}{rs}$$ with common altitude AD = 2.

Now juxtaposing these two triangles we get the oblique triangle ABC, whose sides are

$$a=\frac{[(ps\pm qr)(pr\mp qs)]}{pqrs},\ b=\frac{p^2+q^2}{pq},c=\frac{r^2+s^2}{rs},$$

where the upper and lower sign is to be taken according as the triangles ADC and ADB do not or do overlap.

Moreover we can easily verify that

$$a:b=\frac{r^2+s^2}{rs}:\frac{x^2+y^2}{xy}\ \ for\ x=ps\pm qr, y=pr\mp qs,$$

So that $x^2+y^2=(p^2+q^2)(r^2+s^2)$

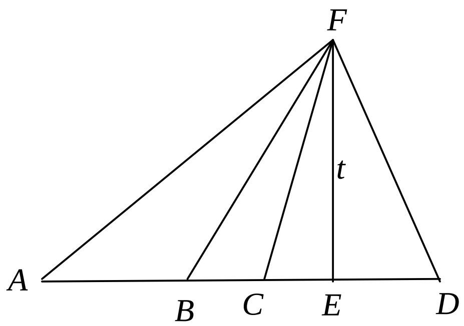
Theorem 2:

Statement:

There exit three triangles with integral sides and area having equal perimeters and areas in the ratio of a: b: c.

Proof:

Let the three required triangles AFB, BFC, CFD standing on the same base AD and of common altitude EF= t



Taking $AF=\left(\frac{p^2+1}{2p}\right)t,\ BF=\left(\frac{q^2+1}{2q}\right)t,\ CF=\left(\frac{r^2+1}{2r}\right)t,\ DF=\left(\frac{s^2+1}{2s}\right)t$

Then $AE=\left(\frac{p^2-1}{2p}\right)t,\ BE=\left(\frac{q^2-1}{2q}\right)t,\ CE=\left(\frac{r^2-1}{2r}\right)t,\ DE=\left(\frac{s^2-1}{2s}\right)t.$

$$\therefore\ AB = AE - BE = \left[\frac{p^2-1}{2p} - \frac{q^2-1}{2q}\right]t,\ BC = BE - CE = \left[\frac{q^2-1}{2q} - \frac{r^2-1}{2r}\right]t$$

$$\text{and } CD = CE + ED = \left[\frac{r^2-1}{2r} - \frac{s^2-1}{2s}\right]t.$$

From the statement, all the three triangles have same perimeters so
$AB + BF + AF = BF + BC + CF = CF + CD + DF$

$$\Rightarrow p + \frac{1}{q} = q + \frac{1}{r} = s + r \tag{1}$$

$$\Rightarrow \frac{q^2-1}{q} = p - \frac{1}{r}, \tag{2}$$

$$\& \frac{r^2-1}{r} = q - s \tag{3}$$

$$\& p - r = s - \frac{1}{q} \tag{4}$$

Again from the statement,

$$\frac{a}{\Delta AFB} = \frac{b}{\Delta BFC} = \frac{c}{\Delta CFD}$$

$$\Rightarrow \frac{a}{AB} = \frac{b}{BC} = \frac{c}{CD} \quad \text{[As height is common to all the triangles]}$$

$$\Rightarrow \frac{a}{\frac{t}{2}\left[\frac{p^2-1}{p} - \frac{q^2-1}{q}\right]} = \frac{b}{\frac{t}{2}\left[\frac{q^2-1}{q} - \frac{r^2-1}{r}\right]} = \frac{a}{\frac{t}{2}\left[\frac{r^2-1}{r} - \frac{s^2-1}{s}\right]}$$

$$\Rightarrow \frac{arp}{p-r} = \frac{b}{p-r} = \frac{c}{\frac{q}{s}\left(s - \frac{1}{q}\right)} \tag{5}$$

$$\Rightarrow \frac{arp}{p-r} = \frac{b}{p-r} = \frac{sc}{q(p-r)} \tag{6}$$

$$arp = b = \frac{sc}{q} \qquad (\text{as } p \neq r)$$

$$\Rightarrow p = \frac{b}{ar},\ s = \frac{bq}{c} \tag{7}$$

From (5) and (7) we have

$$\frac{r^2-1}{r} = q - s = q - \frac{bq}{c} = \frac{q(c-b)}{c}$$

$$\Rightarrow q = \left(\frac{c}{c-b}\right)\left(\frac{r^2-1}{r}\right) \tag{8}$$

Eliminating p, q and s from (2), (7) and (8), we get

$$\left(\frac{c}{c-b}\right)^2\left(\frac{r^2-1}{r}\right)^2-1=\left(\frac{c}{c-b}\right)\left(\frac{r^2-1}{r}\right)\left(\frac{b}{ar}-\frac{1}{r}\right)$$

$$\Rightarrow r^4-\left(d^2+dc+2\right)r^2+de+1=0 \tag{9}$$

where, $d=\dfrac{c-b}{c}$, $e=\dfrac{b-a}{a}$

Now choosing integral values of a, b, c so that the biquadratic in r gives a rational root, then we can easily see that p, q & s are rationals. Thus we obtained the required triangles.

Example:

For example taking a = 2, b = 7, c = 15, we get the rational root r = 5/3.

Thus

$$p=\frac{21}{10},\ q=2, s=\frac{15}{14}$$

$$\therefore AF=\left(\frac{541}{420}\right)t, BF=\left(\frac{5}{4}\right)t, CF=\left(\frac{17}{15}\right)t, DF=\left(\frac{421}{421}\right)t,$$

Taking t = 420, we get

$$AF=541, BF=525, CF=476, DF=421, AB=26, BC=9, CD=195$$

The perimeters of the triangle are same, which is equal to 1092.

Problem (1)

If the sides and area of the triangle are integers, then the area is divisible by 6.

Proof:

If p, q, r, s all are integers then in any rational triangle, the sides are proportional to the numbers

$$\frac{(ps\pm qr)(pr\mp qs)}{pqrs}, \frac{p^2+q^2}{pq}, \frac{r^2+s^2}{rs}$$

Thus the sides of the rational triangles are taken as

a= (ps+qr)(pr-qs), b=rs(p^2+q^2), c=pq(r^2+s^2)

Let 2t be the perimeter of the rational triangle, then

$$t=\frac{a+b+c}{2}.$$

The area of the triangle is given by

$$\Delta=\sqrt{t(t-a)(t-b)(t-c)}=pqrs(ps+qr)(pr-qs) \tag{10}$$

If p, q, r, s are all odd integers or at least one of them is even then $\Delta\equiv 0$ (mod 2).

If one of p, q, r, s is odd then $\Delta\equiv 0$ (mod 3).

Now suppose that none of p, q, r, s is divisible by 3, then

$$(ps+qr)(pr-qs)=rs(p^2-q^2)+pq(r^2-s^2)\equiv\left[\left\{rs(1-1)+pq(1-1)\bmod(3)\right\}\right]$$
$$(ps+qr)(ps-qr)\equiv 0\,(\bmod 3)$$

Because the square of any integer which is not divisible by 3, is congruent to 1 (mod 3).

Hence it follows that $\Delta = pqrs\,(ps+qr)(ps-qr)\equiv 0\,(\bmod 3)$.

Then, $\Delta \equiv 0\left(\bmod 6\right)$.

Theorem (3)
Statement:
There exit triangles whose sides are consecutive integers.

Proof:
Let x – 1, x, x +1 be the sides of a rational triangles, then the area is given by

$$\Delta = \sqrt{t\left(t-a\right)\left(t-b\right)\left(t-c\right)}$$

where $t=\dfrac{3x}{2}=$ half of the perimeter of the triangle.

$$\Delta = \sqrt{\frac{3}{4}\left(\frac{x}{2}\right)^2\left(x^2-4\right)}$$

Since the area of the triangle is rational hence we must have

$$x^2-3y^2=4 \tag{11}$$

Here x and y both cannot be odd simultaneously since $1-4\equiv 3$ (mod 8) is impossible.

Also x and y cannot be of opposite parity since $0-4\equiv 3$ (mod 4), $1-4\equiv 3$ (mod 4) each of which is impossible hence x and y both must be even.

Let us write x = 2u & y = 2v then from (11) we get

$$u^2-3v^2=1 \tag{12}$$

Which is a Pell's equation and whose fundamental solution is u = 2, v = 1.

Hence all solutions of Pell's equation (12) are given by

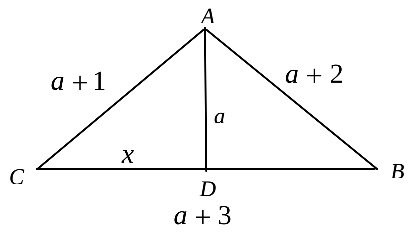
$$u+\sqrt{3}v=\left(2+\sqrt{3}\right)^r,\ r=1,2,3,.......$$

$$\mathit{Thus}\ (u,v)=(2,1),(7,4)............$$

and the required triangles are (3, 4, 5), (13, 14, 15)………………

Problem (2)
The triangle with the altitude 12 and sides 13, 14 15 is the only one triangle in which the altitude and sides are consecutive integer.

Proof:
Let a be the altitude and a + 1, a + 2, a + 3 be the three sides of a triangles ABC.



If the perpendicular is drawn from A to the side a + 3 and x is the base of one of the right triangle formed after drawing perpendicular, then

(I.) $a^2 + x^2 = (a+1)^2$, $a^2 + (a+3 \mp x)^2 = (a+2)^2$

For other perpendiculars, we have

(II) $a^2 + x^2 = (a+2)^2$, $a^2 + (a+1 \mp x)^2 = (a+3)^2$

(III) $a^2 + x^2 = (a+3)^2$, $a^2 + (a+2 \mp x)^2 = (a+1)^2$

Subtracting the result of (I.), we get

2a + 3 $\equiv$ 0 (mod (a - 3)) which is impossible

Similarly (II) gives

$2a + 5 \equiv 0 (\bmod (a+1))$ which is impossible only when a = 2.

Now substituting the two equations of case (III), we get

$\pm 2x - (a+2) = 4$

$2x = a + 6$ (Negative x is inadmissible)

Thus from the first Equation, we get

$$a^2 + \left(\frac{a}{2} + 3\right)^2 = (a+3)^2 \Rightarrow a = 12$$

Hence the theorem follows.

Theorem (4)
Statement:
If in any primitive rational triangle, two sides are odd, the last side > 2 and the difference between the sum of two smaller sides and the larger sides is not unity.

Proof:
Let a, b, c be the integral sides of a triangle with (a, b, c) = 1 and let b < a < c.

(I.) Let us first prove two of a, b, c are odd.

The area of the triangle is given by

$$\Delta = \sqrt{t(t-a)(t-b)(t-c)}, \text{ where } t = \frac{a+b+b}{2}$$

$$= \sqrt{\frac{(a+b+b)}{2}\frac{(b+c-a)}{2}\frac{(a+b-c)}{2}\frac{(c+a-b)}{2}}$$

$$16\Delta^2 = (a+b+c)(b+c-a)(a+b-c)(c+a-b)$$
$$= \left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right] \quad (13)$$

$16\Delta^2 \equiv (1-0)(0-1) \equiv -1$ (mod 4) if a, b and are even.

$16\Delta^2 \equiv (1-0)(0-1) \equiv -1$(mod 4) if a, c even and b odd.

$16\Delta^2 \equiv (1-0)(0-1) \equiv -1$(mod 4) if a, b even and c odd.

Hence two of a, b, c must be odd.

(II) Now let us prove that b > 2

We have $b^2 + c^2 \geq 2bc$ & $a^2 + b^2 \geq 2bc$ *as* $AM \geq GM$

In one of the above two inequality $b^2 \geq c(2b-c)$

Hence $2b \geq c \Rightarrow b \geq c/2$

Similarly from the second inequality $b \geq a/2$

Thus $2b \geq \dfrac{c+a}{2}$ (adding two results) $\Rightarrow b \geq \dfrac{c+a}{4}$

If b = 1, the least value of a = 2 and least value of c = 3 and so $1 > \frac{5}{4}$ which is impossible.

If b = 2, the least value of a = 3 and least value of c = 4 and $2 > \frac{7}{4}$ which is impossible.

If b = 3, the least value of a = 4 and $3 > \frac{9}{4}$ which is impossible.

Hence, b > 2.

(III) Here to prove that $b + a - c \neq 1$.

If possible, suppose b + a – c = 1, then from (13)

$$16\Delta^2 = (a+b+c)(b+c-a)(a+b-c)(c+a-b)$$
$$= (a+b+c)(c+a-b)(c-a+b)$$
$$16\Delta^2 = (2a+2b-1)(2a-b)(2b-1) \quad \text{I} \quad (14)$$

The relation (14) does not hold because left side of (14) is even whereas right side is odd. Thus b + a – c = 1 is impossible.

Hence, b + a – c$\neq$1.

Conclusion:

All the propositions and theorems are the tools for number system and for the theory of numbers. With the help of these propositions and theorems many unsolved problems of rational triangles can be solved.

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