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A Remark on Common Fixed Point of Pairs of Coincidentally Commuting Mappings in D-Metric Spaces

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ABSTRACT

In this paper we prove common fixed point of pairs of coincidentally commuting mappings in D-metric spaces.

Keywords and Phrases: Common fixed point, D-metric space, commuting mappings, coincidentally commuting mapping.

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1. INTRODUCTION

Dhage (1, 2, 3) introduced the concept of D-metric space and proved several results. Rhoades [9] also established interesting result on D-metric spaces. Jungck [4, 5] introduced a more general concept known as compatible mapping in metric spaces. Ume [12] proved non convex minimization theorem in D-metric space. Naidu et. al. [7] developed the concept of balls in a D-metric space. In this direction Sayyed et. al. [10] and kadam[6]proved fixed point theorems in D-metric space. Recently Rathi and Rani[8]proved common fixed point theorem in D-metric space via altering distances between points.

Definition 1. 1: if B (X) is the collection of all non-empty bounded subsets of a D-metric space (X, D) and for A, B, C $\in$ B (X). Let $H(A,B,C) =$ sup $\{D(a,b,c) : a \in A, b \in B, c \in c\}$, then

(i) $H(A,B,C) \geq 0$ and $H(A,B,C) = 0$ implies $A = B = C$ with a singleton further if A = B = C, then $H(A,B,C) =$ perimeter of the largest triangle contained in the set A > 0 otherwise A is singleton,

(ii) $H(A,B,C) = H(B,C,A) = H(C,A,B)$,

(iii) $H(A,B,C) \geq H(A,B,E) + H(A,E,C) + H(E,B,C)$

Definition 1. 2: A point $p \in X$ is said to be fixed point of the multifunction T if $p \in Tp$.

2. COINCIDENTALLY COMMUTING MAPPINGS

The commutativity of pairs of maps is vital for proving the common fixed point theorems and Jungck [4] first used it in the ordinary metric space case.

Definition 2. 1: Two maps $f, g : X \to X$ are said to be commutative or commuting if $(fg)(x) = (gf)x$ for all $x \in X$.

In an ordinary metric space (X,d), Sessa [11] first introduced a weaker version of the commutatively for a pair of selfmaps of *X* as follow.

Definition 2. 2: Two maps $f, g : (X,d) \to (X,d)$ are called weakly commutative or weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

It is shown in Sessa [11] that a weakly commuting pair of maps in metric space is commuting, but the converse may not be true.

In the following we list a few more weaker versions of the commutativity for pairs of maps in metric spaces recently appeared in the literature.

Definition 2. 3: (Jungck [5]) Two maps $f, g : (X,d) \to (X,d)$ are said to be compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in *X* satisfying $\lim_n d(fx_n, gx_n) = 0$. It has been shown in Jungck [5] that every weakly commuting pair of maps is compatible, but the reverse implication may not hold.

Definition 2. 4: Two maps $f, g : X \to X$ are said to be coincidentally commuting or coincidence preserving if they commute at coincidence points.

Obviously every compatible pair of maps is coincidentally commuting but the converse may not be true. Thus we have a one way implication namely. Commuting maps $\Rightarrow$ weakly commuting maps $\Rightarrow$ compatible maps $\Rightarrow$ coincidentally commuting maps.

There are pairs of maps which may have more than one coincidence point, but which do not commute at all such coincidence points.

Example 2. 1: Let $X = R$ and defined *f*, *g*: $R \to R$ by $f(x) = \dfrac{x}{2}$ and $g(x) = x^2$ for $x \in R$. Clearly there are two coincidence points for the maps *f* and *g* in *R* namely 0 and 1/2. Note that *f* and *g* commute at 0, i. e. $(fg)(0) = (gf)(0)$, but

$(fg)(1/2) = 1/8 \neq (gf)(1/2)$, and so *f* and *g* are not coincidentally commuting on *R*.

3. FIXED POINT THEOREMS

We need the following useful lemma in the sequel.

Lemma 3. 1 (D-Cauchy principle): Let $\{y_n\}$ be a sequence in a D-metric space *X* and *D*-bound *k* satisfying.

$$D(y_n, y_{n+1}, y) \leq \lambda^n k$$

for all $m > n \in N, 0 \leq \lambda < 1$. Then $\{y_n\}$ is D-Cauchy. A slight generalization of lemma 3. 1 is Lemma 3. 2.

Lemma 3. 2: Let $\{y_n\}$ be bounded sequence in a D-metrix space *X* with D-bound *k* satisfying

$$D(y_n, y_{n+1}, y_m) \leq \phi^n(k)$$

for all $m > n \in N,$ where ϕ: $R^+ \to R^+$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$. Then $\{y_n\}$ is D-Cauchy.

Let ϕ denote the class of all real function $\phi : R^+ \to R^+$ satisfying the following properties:

(i) ϕ is continuous

(ii) ϕ is nondecreasing

(iii) $\phi(t) < t$ if $t > 0$.

(iv) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$.

A member of the class ϕ is called control function and commonly used as $\phi(t) = \alpha t, 0 \leq \alpha < 1$.

The following lemma are known in the literature:

Lemma 3. 3 (Dhange [3]): If $\phi \in \Phi,$ then $\lim_n \phi^n(0) = 0, n \in N.$

Now we prove the main result of this paper.

Theorem 3. 1: Let $f, g : X \to x$ be two mappings satisfying

$$D(fx, fy, fz) \leq \phi\left[\alpha\left(\frac{(D(gx, fx, gz)^2 + D(gy, fy, gz)^2}{D(gx, fx, gz) + D(gy, fy, gz)}\right) + \beta(gx, gy, gz)\right] \quad \textbf{(3. 1. 1)}$$

for all $x, y, z \in X$, where $\phi \in \Phi$ and $0 \leq 2\alpha + \beta < 1$. Further suppose that

(i) $f(X) \subseteq g(X)$,

(ii) $f(X)$ is bounded and $g(X)$ is complete and

(iii) *f* and *g* are coincidentally commuting.

Then *f* and *g* have a unique common fixed point $v \in X$ and if *g* is continuous at *u*, then *f* is also continuous at *u*.

Proof: Let $x_0 = x \in X$ be arbitrary and define a sequence $\{y_n\}$ in *X* by

$$y_0 = gx_0, y_{n+1} = fx_n = gx_{n+1}, n = 0,1..... \tag{3. 1. 2}$$

Clearly the sequence $\{y_n\}$ is well defined, since $f(x) \subseteq g(X)$. If $y_r = y_{r+1}$ for some $r \in N$. Then

$$y_r = fx_{r-1} = fx_r = gx_r = gx_{r+1} = y_{r+1} = u \tag{3. 1. 3}$$

for some $u \in X$. We show that *u* is a common fixed point of *f* and *g*.

Since $fx_r = gx_r$ and *f, g* are coincidentally commuting.

We have

$fu = fgx_r = gfx_r = gu$.

Now,

$$D(fu, gu, u) = D(fu, gfx_r, fx_r)$$
$$= d(fu, fu, fx_r)$$
$$\leq \phi\left[\alpha \frac{D(gu, fu, gx_r)^2 + D(gu, fu, gx_r)^2}{D(gu, fu, gx_r) + D(gu, fu, gx_r)}\right]$$
$$+\beta(gu, gu, gx_r)$$
$$\leq [\alpha D(gu, fu, u) + D(gu, fu, u) + \beta D(gu, gu, u)]$$
$$= \phi(2\alpha + \beta) D(fu, gu, u)$$

which yields that $fu = gu = u$. This shows that *u* is a common fixed point of *f* and *g*. Therefore, we assume that $y_n \neq y_{n+1}$ for all $n \in N$. We show that $\{y_n\}$ is D-Cauchy.

Now for *m* > 1, we have

$$D(y_1, y_2, y_m) = D(fx_0, fx_1, fx_{m-1})$$
$$\leq \phi\left[\alpha\left(\frac{D(gx_0, fx_1, gx_{m-1})^2 + D(gx_1, fx_1, gx_{m-1})^2}{D(gx_0, fx_1, gx_{m-1}) + D(gx_1, fx_1, gx_{m-1})}\right) + \beta D(gx_0, gx_1, gx_{m-1})]\right]$$
$$\leq \phi[\alpha(D(y_0, y_1, y_{m-1}) + D(y_1, y_2, y_{m-1})) + \beta(D(y_0, y_1, y_{m-1}))]$$
$$< \frac{\alpha + \beta}{1 - \alpha} D(y_0, y_1, y_{m-1})$$

Again for $m > 2$, one has

$$D(y_2, y_3, y_m) = D(fx_1, fx_2, fx_{m-1})$$
$$\leq \phi\left[\alpha\left(\frac{D(gx_1, fx_1, gx_{m-1})^2 + D(gx_2, fx_2, gx_{m-1})^2}{D(gx_1, fx_1, gx_{m-1}) + D(gx_2, fx_2, gx_{m-1})}\right) + \beta(D(gx_1, gx_2, gx_{m-1}))\right]$$
$$< \phi\left[\alpha(D(y_1, y_2, y_{m-1}) + D(y_2, y_3, y_{m-1})) + \beta(D(y_1, y_2, y_{m-1}))\right]$$
$$< \left(\frac{\alpha+\beta}{1-\alpha}\right) D(y_1, y_2, y_{m-1})$$
$$< \left(\frac{\alpha+\beta}{1-\alpha}\right)^2 D(y_0, y_1, y_{m-2}).$$

Similarly, in general for $m > n$, one has

$$D(y_n, y_{n+1}, y_m) < \left(\frac{\alpha+\beta}{1-\alpha}\right)^n D(y_0, y_1, y_{m-n}).$$

Now an application of Lemma 3. 1 yields that $\{y_n\}$ is D-Cauchy. Since $g(X)$ is complete, there is a point $u \in g(X)$ such that $\lim_n y_n = u$, i. e. $\lim_n fx_n = \lim_n gx_n = u$. We prove that u is a common fixed point of f and g.

Since $u \in g(X)$ there is a point p □ X such that $gp = u$. First show that $fp = gp = u$.

Now

$$D(fp, gp, gp) = \lim_n D(fp, fx_n, fx_n)$$
$$\leq \lim_n \phi\left[\alpha\left(\frac{D(gp, fp, gx_n)^2 + D(gx_n, fx_n, gx_n)^2}{D(gp, fp, gx_n) + D(gx_n, fx_n, gx_n)}\right) + \beta(D(gp, gx_n, gx_n)\right]$$
$$\leq \phi\left[\alpha(D(gp, fp, gp) + 0) + \beta 0\right]$$
$$= \phi(\alpha D(fp, gp, gp))$$

which implies that $fp = gp$ since $\phi \in \Phi$.

Then proceeding with the arguments similar to that in case I. it is proved that $u = fp = gp$ is a common fixed point of f and g.

To prove the uniqueness, let v ($\neq u$) be another common fixed point of f and g, then

$$D(u, u, v) = D(u, v, u)$$
$$= D(fu, fv, fu)$$
$$\leq \phi\left[\alpha\left(\frac{D(gu, fu, gu)^2 + D(gv, fv, gu)^2}{D(gu, fu, gu) + D(gv, fv, gu)}\right) + \beta(gu, gv, gu)\right]$$
$$\leq \phi\left[\alpha\ D(v, v, u) + \beta(u, v, u)\right]$$

$$D(u,u,v) = D(u,v,u) < \frac{\alpha}{(1-\beta)} D(v,v,u) \qquad \textbf{(3. 1. 4)}$$

Again interchanging the role of *u* and *v* we obtain

$$D(v,v,u) < \frac{\alpha}{1-\beta} D(u,u,v) \qquad \textbf{(3. 1. 5)}$$

Which is contradiction and hence $u = v$.

Finally we prove the continuity of *f* at *u*. Let $\{z_n\}$ be any sequence converging to the common fixed point *u* of *f* and *g*. Then by definition of the D-Convergence, we have

$$\lim_{m,n} D(z_m, z_n, u) = 0$$

Now by (3. 1. 1)

$$D(fz_m, fz_n, fu) \le \phi\left[\alpha\left(\frac{D(gz_m, fz_m, gu)^2 + D(gz_n, fz_n, gu)^2}{D(gz_m, fz_m, gu) + D(gz_n, fz_n, gu)}\right) + \beta(gz_m, gz_n, gu)\right]$$

Therefore

$$\lim_{nm} D(fz_m, tz_n, fu) \le \phi\left[\alpha \lim_m (D(u, fz_m, u) + \lim_m D(u, fz_n, u)\right]$$

But

$$\lim_m D(u, fz_m, u) = \lim_m D(fu, fz_m, fu)$$

$$\le \lim_m \phi\left[\alpha\left(\frac{D(gu, fu, gu)^2 + D(gz_m, fz_m, gu)^2}{D(gu, fu, gu) + D(gz_m, fz_m, gu)}\right) + \beta D(gu, gz_m, gu)\right]$$

$$\le \phi(\lim_m D(u, fz_m, u)) \qquad \textbf{(3. 1. 6)}$$

which implies that

$$\lim_m D(u, fz_m, u) = 0\,.$$

Similarly

$$\lim_n D(u, fz_n, u) = 0$$

It follows that

$$\lim_{m,n} D(fz_m, fz, fu) = 0\,.$$

and so *f* is continuous at *u*. This completes the proof.

Corollary 3. 1: Let *f*, *g*,: $X \to X$ be two mappings satisfying

$$D(fx, fy, fz) \le \alpha(D(gx, gy, gz)) \qquad \textbf{(3. 1. 7)}$$

for all $x, y, z \in X$, where $0 \le \alpha < 1$. Further if the conditions (i) – (iii) of theorem 3. 1 hold, them *f* and *g* have a unique common fixed point $u \in X$ and if *g* is continuous at *u*, then so is also *f*.

Corollary 3. 2: Let *f, g*: $X \to X$ be two mappings m. p the positive integers, satisfying

$$D(f_x^p, f_y^p, f_z^p) \leq \phi\left[\alpha\left(\frac{D(g_x^m, f_x^p, g_z^m)^2 + D(g_y^m, f_y^p, g_z^m)^2}{D(g_x^m, f_x^p, g_z^m) + D(g_y^m, f_y^p, g_z^m)}\right) + \beta(g_x^m, g_y^p, g_z^m)\right] \quad \textbf{(3. 8)}$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Further suppose that

(i) $f^p(X) \subseteq g^m(X)$.

(ii) $f^p(X)$ is bounded and $g^m(X)$ is complete, and

(iii) *f* and *g* are commuting.

Then *f* and *g* have a unique common fixed point $u \in X$ and if g^m is continuous at *u*, then f^p is also continuous at *u*.

Proof: By theorem 3. 1 f^p and g^m have a unique common fixed point *u*. Then by hypothesis (ii), we obtain

$$f^p(fu) = f(f^p u) = fu = f(g^m u) = g^m(fu).$$

This shows that *fu* is again a common of t^p and g^m . By uniqueness of *u*, we get *fu* = *u*. Similarly, *gu* = *u*. Thus *f* and *g* have a unique common fixed point. This complete the proof.

Corollary 3. 3: Let *f, g*: $X \to X$ be two mappings satisfying

$$[D(fx, fy, fz)]^2 \leq \phi\left[\alpha\left(\frac{D(gx, fx, gz)^2 + D(gy, fy, gz)^2}{D(gx, fx, gz) + D(gy, fy, gz)}\right) + \beta D(gx, gy, gz)^2\right] \quad \textbf{(3. 9)}$$

for all $x, y, z \in X$, $0 \leq \alpha \leq 1$. Suppose further that the hypothesis (i) – (iii) of theorem 3. 1 hold. Then *f* and *g* have a unique fixed point $u \in X$ and if *g* is continuous at *u* then *f* is also continuous at *u*.

Proof: Since for any $a \leq 0, b \geq 0$, we have $ab \leq \max\ \{a^2, b^2\}$, an so condition (3. 9) implies condition (3. 4) with $\phi\ (t) = \sqrt{\alpha t}$. Now the desired conclusion follows by an application of theorem 3. 1. The proof is complete.

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