International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 1 (2014), pp. 37-43

http://www.irphouse.com

$N(k)$ −Contact Metric Manifold Admitting Semi-Symmetric Metric Connection

V. S. Prasad, C. R. Premalatha and H. G. Nagaraja

Department of Mathematics, Regional Institute of Education, Mysore-570 006, India
E-mail address: tvsrie@yahoo.co.in
Department of Mathematics, Bangalore University,
Central College Campus, Bengaluru – 560 001, India
E-mail address: hgnraj@yahoo.com, premalathacr@yahoo.co.in

ABSTRACT

We study $N(k)$-contact metric manifolds admitting semi-symmetric metric connection. We obtain the scalar curvature with respect to semi-symmetric metric connection of $N(k)$ −contact metric manifold by appyling certain curvature conditions.

2010 Mathematics Subject Classification: 53C25, 53D15.

Key words and phrases. N(k)-contact metric manifolds, semi-symmetric metric connection, flat, -flat, Ricci-symmetry, ϕ-recurrent, M-projective curvature tensor, concircular curvature tensor, Einstein.

1. INTRODUCTION

The notion of k −nullity distribution of Riemannian manifolds was introduced by S. Tanno in 1988 [8]. The contact metric manifold where the characteristic vector field ξ belongs to the k-nullity distribution is called N(k) −contact metric manifold. Blair [5] and many others extensively studied N(k) −contact metric manifolds. Friedman and Schouten [1] introduced the notion of semi-symmetric linear connection in a differentiable manifold and a systematic study of semi-symmetric metric connection on a Riemannian manifold was given by Yano[2] in 1970. Semi-symmetric metric connections on Riemannian manifolds are studied by authors [3]. The semi-symmetric metric connection plays an important role in the study of Riemannian manifolds involving physical problems. This motivates us to study such connections in N(k) −contact metric manifolds.

This paper is organized as follows. We give some preliminary results in Section 2. In section 3 we present a brief information of $N(k)$ −contact metric manifolds admitting semi-symmetric metric connection.

In sections 4 and 5, we study M-projective curvature tensor and concircular curvature tensors on $N(k)$ −contact metric manifolds with respect to semi-symmetric metric connection by considering flat, Ricci-symmetry and ϕ-recurrent conditions. The following table summarizes the results proved in this paper.

Condition	Result
$\widetilde{W}^*(X,Y)\xi = 0$	$k = \frac{2}{1-n}$ and $\tilde{r} = \frac{2n(n+1)}{1-n}$
$\widetilde{W}^*.\tilde{S} = 0.$	$k = \frac{4n^2-4n+1}{2n(4n-3)}$ and $\tilde{r} = \dfrac{-8n^2+2n+1}{(4n-3)}$
ϕ − recurrent	$k = \frac{4n-1}{2n}$ and $\tilde{r} = 2n - 1$

2. PRELIMINARIES

In this section, some general definitions and basic formulas are presented which will be used later. A $(2n+1)$ dimensional C^∞-differentiable manifold M is said to admit an almost contact metric structure (ϕ, ξ, η, g) if the following relations hold. [6].

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi)=1, \eta\xi = 0, \eta o \phi = 0. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \phi X) = 0, g(X, \xi) = \eta(X) \tag{2.3}$$

where ϕ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on M. A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies

$$g(X, \phi Y) = d\eta(X,Y) \tag{2.4}$$

for all vector fields X, Y. The (1, 1) tensor field h defined by $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie differentiation, is a symmetric operator and satisfies

$$h\phi = \phi h, trh = tr\phi h = 0, h\xi = 0. \tag{2.5}$$

Further we have [5],

$$\nabla_X \xi = -\phi X - \phi h X, (\nabla_X \eta)Y = g(X + hX, \phi Y) \tag{2.6}$$

where r denotes the Riemannian connection of g.

The k-nullity distribution $N(k)$ of a contact metric manifold M [4] is defined by

$$N(k): p \to N_p(k,\mu) = \{Z \in T_p(M): R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \tag{2.7}$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then the contact metric manifold is called a $N(k)$-contact metric manifold [7].

In a $(2n+1)$-dimensional $N(k)$-contact metric manifold, the following relations hold. [8, 9, 10]:

$$h^2 = (k-1)\phi^2, k \leq 1 \tag{2.8}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.9}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \tag{2.10}$$

$$S(X,Y) = 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) + [2(1-n) + 2nk]\eta(X)\eta(Y) \tag{2.11}$$

$$S(X,\xi) = 2nk\eta(X) \tag{2.12}$$

$$r = 2n(2n-2+k) \tag{2.13}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y) \tag{2.14}$$

$$(\nabla_X\eta)(Y) = g(X + hX, \phi Y) \tag{2.15}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{2.16}$$

where Q, S and r are respectively the Ricci operator, the Ricci tensor and the scalar curva-ture of M^{2n+1}.

The M-projective curvature tensor W^* in M^{2n+1} is given by [12]

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{2.17}$$

3. $N(k)$-CONTACT METRIC MANIFOLD M^{2n+1} ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

Let $\tilde{\nabla}$ be a linear connection on $(2n+1)-$ dimensional differential manifold M^{2n+1}. The torsion tensor T is given by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y].$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor vanishes. If $\tilde{\nabla}g = 0$, where g is a Riemannian metric on M^{2n+1}, then connection $\tilde{\nabla}$ is called a metric connection[2]. A linear connection is said to be a semi-symmetric connection in a Riemannian manifold if its torsion tensor

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{3.1}$$

where π is a 1-form defined by $\pi(X) = g(X,\rho)$ and ρ is a vector field.

A semi-symmetric metric connection in an almost contact metric manifold is defined by

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$$

where $\eta(Y) = g(Y,\xi)$.

A relation between the semi-symmetric connection $\tilde{\nabla}$ and Levi-Civita connection ∇ of M^{2n+1} is given by [3].

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) - g(X,Y)\xi, \eta(Y) = g(Y,\xi).$$

Further a relation between the curvature tensor R and $\tilde{R}$ of type (1, 3) of the connections ∇ and $\tilde{\nabla}$ respectively [3] is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - L(Y,Z)X + L(X,Z)Y - g(Y,Z)FX + g(X,Z)FY, \tag{3.2}$$

where L is a tensor field of type (0; 2) given by

$$L(Y,Z) = (\nabla_Y\eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}g(Y,Z) = (\overline{\nabla}_Y\eta)(Z) - \frac{1}{2}g(Y,Z), \tag{3.3}$$

and F is a tensor field of type (1, 1) given by $g(FY,Z) = L(Y,Z)$ for any vector fields Y, Z.

$$\overline{R}(X,Y)Z) = R(X,Y)Z - g(Y + hY, \phi Z)X + g(X + hX, \phi Z)Y + g(Y,Z)(\phi X + \phi hX) - g(X,Z)(\phi Y + \phi hY) + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi - g(Y,Z)X + g(X,Z)Y. \qquad (3.4))$$

From (2.10) and (3.4), we have

$$\bar{R}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \eta(Y)(\phi X + \phi hX) - \eta(X)(\phi Y + hY) \qquad (3.5)$$
$$\bar{R}(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi] + g(X + hX, \phi Y)\xi + \eta(Y)(\phi X + \phi hX), \qquad (3.6)$$
$$\bar{R}(\xi,X)\xi = k[\eta(Y)\xi - Y] - (\phi Y + \phi hY), \qquad (3.7)$$

On contracting (3.4), we get

$$\bar{S}(Y,Z) = S(Y,Z) - (2n-1)[g(Y + hY, \phi Z) - \eta(Y)\eta(Z) + g(Y,Z)] \qquad (3.8)$$

where $\bar{S}$ and S are Ricci tensors of the connections $\bar{\nabla}$ and ∇ respectively.

$$\bar{Q}(Y) = QY - (2n-1)[-(\phi Y + \phi hY) - \eta(Y)\xi + Y], \qquad (3.9)$$

whwhere $\bar{Q}$ and Q are Ricci operators of the connections $\bar{\nabla}$ and ∇ respectively.

$$\bar{S}(Y,\xi) = 2nk\eta(Y). \qquad (3.10)$$

Again contracting (3.8) over Y, Z, we get

$$\bar{r} = r - (2n-1)2n, \qquad (3.11)$$

By virtue of (2.15), (3.11) yields

$$\bar{r} = 2n(k-1). \qquad (3.12)$$

where $\bar{r}$ and r are scalar curvatures of the connections $\bar{\nabla}$ and ∇ respectively

4. M –PROJECTIVE CURVATURE TENSOR IN AN $N(k)$-CONTACT METRIC MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

The M –Projective curvature tensor in an $N(k)$ –contact metric manifold $M^\wedge(2n+1)$ with respect to semi-symmetric metric connection is given by

$$\bar{W}^*(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{4n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \qquad (4.1)$$

From(3.5), (3.6), (3.8) and (4.1), we have

$$\bar{W}^*(X,Y)\xi = \frac{k}{2}[\eta(Y)X - \eta(X)Y] + \eta(Y)(\phi X + \phi hX) - \eta(X)$$
$$(\phi Y + \phi hY) - \frac{1}{4n}[\eta(Y)\bar{Q}X - \eta(X) \qquad (4.2)$$
$$\bar{W}^*(\xi,Y)Z = -k[\eta(Z)Y - g(Y,Z)\xi] - g(Y + hY, \phi Z)$$
$$\xi - \eta(Z)(\phi Y + \phi hY) + \frac{1}{4n}[2nk\eta(Z)Y - \bar{S}(Y,Z)\xi + \eta(Z)\bar{Q}Y - 2nkg(Y,Z)\xi], (4.3)$$
$$\eta(\bar{W}^*(\xi,Y)Z) = -k[\eta(Z)\eta(Y) - g(Y,Z)] - g(Y + hY, \phi Z)$$
$$+\frac{1}{4n}[4nk\eta(Z)\eta(Y) - \bar{S}(Y,Z) - 2nkg(Y,Z)] \qquad (4.4)$$

and

$$\eta(\bar{W}^*(X,Y)Z) = \frac{k}{2}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - \frac{1}{4n}[\bar{S}(Y,Z)\eta(X) - \bar{S}(X,Z)\eta(Y)]$$
$$-g(Y + hY, \phi Z)\eta(X) + g(X + hX, \phi Z)\eta(Y) \qquad (4.5)$$

Definition 1. An $N(k)$ –contact metric manifold M^{2n+1} is

(1)M-projectively flat with respect to semi-symmetric metric connection if

$$\bar{W}^*(X,Y,Z,U) = 0.$$

(2)ξ – M-projectively flat with respect to semi-symmetric metric connection if

$$\bar{W}^*(X,Y)\xi = 0.$$

(3) M-projectively Ricci-symmetric with respect to semi-symmetric metric connection if

$$\bar{W}^*.\bar{S} = 0.$$

(4) M – projective ϕ – recurrent with respect to semi-symmetric metric connection if and only if there exists a 1- form A such that

$$\phi^2\big((\nabla_U\bar{W}^*)(X,Y)Z\big) = A(U)\bar{W}^*(X,Y)Z.$$

Definition 2. A contact metric manifold is said to be (i) Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant, (ii) η -Einstein if $S(X,Y) = \alpha\, g(X,Y) + \beta\,\eta(X)\,\eta(Y)$, where α and β are smooth functions.

Suppose is M^{2n+1} is ξ-M- projectively flat. i.e.

$$\bar{W}^*(X,Y)\xi = 0 \tag{4.6}$$

Then in view of (4.1), we have

$$\bar{R}(X,Y)\xi = \frac{1}{4n}[\bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y + \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y] \tag{4.7}$$

By virtue of (3.5), (3.10) and (4.2), above equation reduces to

$$\frac{k}{2}[\eta(Y)X - \eta(X)Y] + \eta(Y)(\phi X + \phi hX) - \eta(X)$$
$$(\phi Y + \phi hY) = \frac{1}{4n}[\eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y]. \tag{4.8}$$

Putting $Y = \xi$ and using (3.12), (4.17) reduces to

$$\bar{Q}X = 2nk[X - \eta(X)\xi] + 4n(\phi X + \phi hX) + 2nk\eta(X)\xi. \tag{4.9}$$

Contracting the above equation with W, we get

$$\bar{S}(X,W) = 2nk[g(X,W) - \eta(X)\eta(W)]$$
$$+4ng(\phi X + \phi hX, W) + 2nk\eta(X)\eta(W). \tag{4.10}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at any point. Putting $X = W = e_i$ in the above equation and summing over i, $1 \le i \le 2n+1$, we get

$$k = \frac{2}{1-n} \tag{4.11}$$

Then from (3.11), we obtain

$$\bar{r} = \frac{2n(n+1)}{1-n} \tag{4.12}$$

Since $k = 1$ in (4.20) leads to $n = -1$, M^{2n+1} must be non-Sasakian. Thus we can state that

Theorem 4.1 In a ξ-M-projectively flat non-Sasakian N(k)-contact metric manifold admitting semi-symmetric metric connection, we have $k = \frac{2}{1-n}$ and $\bar{r} = \frac{2n(n+1)}{1-n}$.

Since a M-projectively flat N(k)-contact metric manifold is always ξ-M-projectively flat, we have

Corollary 4.1. In an M-projectively flat non-Sasakian N(k)-contact metric manifold admitting semi-symmetric metric connection, we have $k = \frac{2}{1-n}$ and $\bar{r} = \frac{2n(n+1)}{1-n}$.

Suppose $M^{2n+1}$is M-projectively Ricci-symme tric. i.e.

$$\bar{W}^*.\bar{S} = 0. \tag{4.13}$$

In this case, we can write

$$\bar{S}(\bar{W}^*(U,X)Y,Z) + \bar{S}(Y,\bar{W}^*(U,X)Z) = 0. \tag{4.14}$$

Taking $U = Z = \xi$ in (4.14) and using (3.10), (3.12) and (4.3), we have

$$nk^2g(X,Y) + \frac{k(2nk+1)}{2}\eta(X)\eta(Y) - \frac{4nk+1}{4n}\bar{S}(X,Y)$$

$$+\frac{(2n+1)}{4n}\bar{S}(\phi X+\phi hX,Y)+\frac{2(n-1)}{4n}\bar{S}(hX,Y)=0. \quad (4.15)$$

Replacing Y by ϕY in (4.15), we obtain

$$nk^2 g(X,\phi Y)-\frac{4nk+1}{4n}\bar{S}(X,\phi Y)-\frac{2n+1}{4n}\bar{S}(\phi X+\phi hX,\phi Y)$$
$$+\frac{2(n-1)}{4n}\bar{S}(hX,\phi Y)=0. \quad (4.16)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at any point. Setting $X = Y = e_i$ in the above equation and summing over, $1\le i\le 2n+1$, we obtain

$$k=\frac{4n^2-4n+1}{2n(4n-3)}. \quad (4.17)$$

Then from (3.11), we obtain

$$\bar{r}=\frac{-8n^2+2n+1}{4n-3} \quad (4.18)$$

Thus we have

Theorem 4.2. In an M −projectively Ricci-symmetric $N(k)$-contact metric manifold admitting semi-symmetric metric connection, k and $\bar{r}$ are given by (4.17) and (4.18) respectively.

Suppose M^{2n+1} is ϕ-recurrent, then we have

$$-(\nabla_U\bar{W}^*)(X,Y)Z+\eta\big((\nabla_U\bar{W}^*)(X,Y)Z\big)=A(U)\bar{W}^*(X,Y)Z. \quad (4.19)$$

Contracting (4.19) with ξ, we obtain

$$A(U)\eta(\bar{W}^*(X,Y)Z)=0. \quad (4.20)$$

Since A is a non-zero 1-form, we have

$$\eta(\bar{W}^*(X,Y)Z)=0. \quad (4.21)$$

Using (4.5), the above equation yields

$$\frac{K}{2}[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]-\frac{1}{4n}[\bar{S}(Y,Z)\eta(X)-\bar{S}(X,Z)\eta(Y)]$$
$$-g(Y+hY,\phi Z)\eta(X)+g(X+hX,\phi Z)\eta(Y)=0 \quad (4.22)$$

Taking $X=\xi$ in (4.22) and using (3.10), we get

$$\bar{S}(Y,Z)=2nkg(Y,Z)-4ng(Y+\phi Y,\phi Z). \quad (4.23)$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at any point. Taking $Y=Z=e_i$ in the above equation and summing over i, $1\le i\le 2n+1$, we get

$$k=\frac{4n-1}{2n} \quad (4.24)$$

Plugging this in (3.11), we obtain

$$\bar{r}=2n-1 \quad (4.25)$$

Thus we state that

Theorem 4.3. In an M-projectively ϕ-recurrent N(k)-contact metric manifold with respect to semi-symmetric metric connection, we have $k=\frac{4n-1}{2n}$ and $\bar{r}=2n-1$.

REFERENCES

[1] A.Friedmann and J. A. Schouten, Uber die Geometrie der halbsymmetrischen Ubertragungen, Mathematische Zeitschrift, vol. 21, no. 1, pp. 211-223, 1924.

[2] K. Yano, On semi-symmetric metric connection, Revue Roumaine de Mathematiques Pures et Appliquees, vol. 15, pp. 1579-1586, 1970.

[3] G. Pathak and U. C. De, On semi-symmetric metric connection in a Kenmotsu manifold, Bulletin of the Calcutta Mathematical Soceity, vol. 94, pp. 319-324, 2002.

[4] S. Tanno, Ricci curvatures of contact Riemannian manifolds, The Tohoku Mathematical Journal, vol. 40, no.3, pp. 441-448, 1988.

[5] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics, vol. 91, no. 1-3, pp. 189-214, 1995.

[6] D. E. Blair, Contact manifolds in Riemannian Geometry, Lecture Notes in Mathematics, Vol.509, Springer, Berlin, Germany, 1976.

[7] D. E. Blair, J. S. Kim and M. M. Tripathi, On the concircular curvature tensor of a contact metric manifold, J. Korean Math. Soc. Vol. 42, no. 5, pp. 883-892.

[8] Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos, A decomposition of the curvature tensor a of contact manifold satisfying $\mathrm{R}(\mathrm{X},\mathrm{Y})\xi = \mathrm{k}[\eta(\mathrm{Y})\mathrm{X} - \eta(\mathrm{X})\mathrm{Y}]$, Mathematics Technical Report, University of Ioanniana, 1992.

[9] D. E. Blair, Th. Koufogiorgos and R. Sharma, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Mathematical Journal Vol. 13, pp. 391-401, 1990.

[10] D. E. Blair and H. Chen, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, II, Bulletin of the Institute of Mathematics Academia Sinica, vol. 20, pp. 379-383, 1992.

[11] K. Yano, Concircular geometry-I. Concircular transformations, Proceedings of the Japan Academy, vol. 16, pp. 195-200, 1940.

[12] G. P. Pokhariyal and R. S. Mishra, Curvature tensors and their relativistics significance, Yokohama Mathematical Journal, vol. 18, pp. 105-108, 1970.