International Journal of Mathematics Research.
ISSN 0976-5840 Volume 6, Number 1 (2014), pp. 31-35

http://www.irphouse.com

Numerical Evaluation of Differintegrals of Arbitrary Orders

M. Acharya[*], B. P. Acharya[*] and S. B. Sahoo[**]

** ITER, S'O'A University, Bhubaneswar*
[**] *Trident Academy of Creative Technology, BPUT, Odisha, (India)*
(email: milu_acharya@yahoo.com, sundarbadan@gmail.com)

Abstract

A numerical technique has been developed for the evaluation of integrals and derivatives of fractional orders. The technique is based on the Cauchy integral formula of complex analysis and is capable of yielding accuracy of fifteen decimal places. The numerical verification of the technique has been performed in respect of some standard functions.

Subject Classification Primary (1999): 65D 25, 65D 30

Key words: Fractional integrals, Fractional derivatives, Quadrature rules, Cauchy integral formula.

1. Introduction

The reply of Leibnitz to the query of L'Hospital in the year 1695 viz. "$D^n x, D = d/dx$ when $n = 1/2$ is an apparent paradox from which, one day, useful consequences will be drawn" is considered to be the origin of fractional calculus. After extensive research this subject has acquired a solid and strong foundation and a number of excellent texts on this subject are available. It is pertinent to note that the numerous and manifold applications of fractional calculus in diverse branches of science and engineering have justified the later part of the reply of Leibnitz to L'Hospital. Some of these applications have been highlighted by Dalir and Bashour [1], Miller and Ross [2], Oldham and Spanier [3], etc. Fractional integrals and fractional derivatives are also jointly known as differintegrals of arbitrary orders. Out of the various definitions of differintegrals the following are due to Riemann and Liouville (R-L).

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \tag{1}$$

$$D^{\beta}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^x \frac{f(t)}{(x-t)^{1+\beta}} dt \tag{2}$$

where the parameters α and β lie in the open interval $(0,1)$ and D is the derivative operator d/dx.

In view of numerous applications of fractional calculus, some numerical treatment of the subject has also been undertaken by some researchers such as Acharya et al [4], Diethelm and Walz [5] and Lether et al [6] etc. Our object in present paper is to formulate a technique based on the celebrated Cauchy integral formula of complex analysis for the numerical evaluation of differintegrals of arbitrary orders.

2. Formulation of the method

One of the definitions of differintegrals has been motivated in the following form which is founded on the Cauchy integral formula of complex analysis (cf. Miller and Ross [2], Oldham and Spanier[3])

$$D^{q}f(z) = \frac{\Gamma(1+q)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+q}} \tag{3}$$

where C is a closed contour, surrounding the point z, contained in the domain of analyticity of the function $f(\zeta)$, q is not a negative integer and the contour C starts and ends at $\zeta = 0$. It is noteworthy that for non integer values of q the integrand in equation (3) has a branch point at $\zeta = z$. Deforming the contour C into a contour C' lying on both sides of the branch line for $(\zeta - z)^{-q-1}$ and a small loop at z, the differintegral $D^{q}f(z)$ given by equation (3) reduces to the differintegrals (equations (1) and (2)) in the R-L sense. Therefore we have the following equation.

$$\frac{\Gamma(1+q)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+q}} = \frac{1}{\Gamma(-q)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1+q}} d\zeta \tag{4}$$

where $-1 < q < 0$ for fractional integration and $0 < q < 1$ for fractional differentiation.

Let x be a positive real number and the contour γ be a square of side $\sqrt{2}x$ having vertices at the following set of points and with x as its centroid.

$$z_j = x\left(1 - i^{j-1}\right), \qquad j = 1,2,3,4. \tag{5}$$

Replacing the point z by x and the contour C by γ in equation (3) we get the following.

$$D^{q}f(x) = \frac{\Gamma(1+q)}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{(\zeta - x)^{1+q}}. \tag{6}$$

The contour γ can be expressed as $\gamma = \cup_{k=1}^{4} L_k$, where L_k is a directed line segment from z_k to z_{k+1}, $z_5 = z_1$. Thus from equation (6) we have

$$D^q f(x) = \frac{\Gamma(1+q)}{2\pi i}\sum_{k=1}^{4}\int_{L_k}\frac{f(\zeta)d\zeta}{(\zeta - x)^{1+q}}\,. \tag{7}$$

Each of the integrals under the summation sign in equation (7) can be approximated by an open quadrature rule meant for integrals of analytic functions along a directed line segment. Some of these rules can be found in Milovanovic [7]. However, from the points of view of both accuracy and simplicity the n-point transformed Gauss-Legendre rule due to Lether [8] is considered to be quite preferable which is given as follows.

$$R(\varphi, n, L) = H\sum_{k=1}^{n} w_k \varphi(z_0 + Ht_k) \tag{8}$$

where $\varphi(\zeta)$ is an analytic function, L is a directed line segment from $z_0 - H$ to $z_0 + H$, t_k's are the zeros of the Legendre polynomial of degree n and w_k's are the associated coefficients.

The accuracy of the approximation of the integrals along L_k given in the right hand side of equation (7) by any quadrature method is quite likely to be affected due to the presence of the factor $(\zeta - x)^{1+q}$ in the denominator of the integrand. For avoiding this the following approaches for cases $-1 < q < 0$ and $0 < q < 1$ are adopted. We consider the former case first.

$$D^q f(x) = \frac{\Gamma(1+q)}{2\pi i}\sum_{k=1}^{4}\int_{L_k}\frac{f(\zeta) - F(\zeta)}{(\zeta - x)^{1+q}}d\zeta + I_1 + I_2 \tag{9}$$

where $F(\zeta) = f(x) + f'(x)(\zeta - x)$ and I_1 and I_2 are given as follows.

$$\left.\begin{aligned} I_1 &= \frac{f(x)\Gamma(1+q)}{2\pi i}\int_{\gamma}\frac{d\zeta}{(\zeta - x)^{1+q}} = f(x)D^q(1) = \frac{-f(x)}{qx^q\Gamma(-q)},\\ I_2 &= \frac{f'(x)\Gamma(1+q)}{2\pi i}\int_{\gamma}\frac{d\zeta}{(\zeta - x)^{q}} = qf'(x)D^{(q-1)}(1) = \frac{qx^{1-q}f'(x)}{\Gamma(2-q)}.\end{aligned}\right\} \tag{10}$$

Similarly for the later case i.e. $0 < q < 1$ we have

$$D^q f(x) = \frac{\Gamma(1+q)}{2\pi i}\sum_{k=1}^{4}\int_{L_k}\frac{f(\zeta) - G(\zeta)}{(\zeta - x)^{1+q}}d\zeta + J_1 + J_2 + J_3 \tag{11}$$

where $G(\zeta) = f(x) + (\zeta - x)f'(x) + \frac{1}{2}(\zeta - x)^2 f''(x)$ and J_1, J_2 and J_3 are given as follows. It is noteworthy that in case of fractional derivative, for the sake of the desired accuracy the function $G(\zeta)$ is subtracted out from $f(\zeta)$ instead of $F(\zeta)$.

$$\left.\begin{aligned}
J_1 &= \frac{f(x)\Gamma(1+q)}{2\pi i}\int_\gamma \frac{d\zeta}{(\zeta-x)^{1+q}} = f(x)D\{D^{q-1}(1)\} = \frac{x^{-q}f(x)}{\Gamma(1-q)},\\
J_2 &= \frac{f'(x)\Gamma(1+q)}{2\pi i}\int_\gamma \frac{d\zeta}{(\zeta-x)^{q}} = qf'(x)D\{D^{q-2}(1)\} = \frac{qx^{1-q}f'(x)}{\Gamma(2-q)},\\
J_3 &= \frac{f''(x)\Gamma(1+q)}{4\pi i}\int_\gamma \frac{d\zeta}{(\zeta-x)^{q-1}} = \{q(q-1)f''(x)/2\}\,D\{D^{q-3}(1)\}\\
&= \frac{q(q-1)x^{2-q}f''(x)}{2\Gamma(3-q)}.
\end{aligned}\right\} \tag{12}$$

It is noteworthy that for obtaining the expressions for J_1, J_2 and J_3 the fundamental theorem of fractional calculus (cf. Dannon [9]) has been made use of.

Hence the approximation for fractional integral of order $q \in (-1,0)$ and that for fractional derivative of order $q \in (0,1)$ can be respectively furnished in the following forms.

$$D^q f(x) \approx \frac{\Gamma(1+q)}{2\pi i}\sum_{k=1}^{4} R(\psi, n, L_k) + \mathrm{CRI}\,, -1 < q < 0, \tag{13}$$

where

$$\left.\begin{aligned}
\psi(\zeta) &= \frac{f(\zeta)-F(\zeta)}{(\zeta-x)^{1+q}},\\
CRI = I_1 + I_2 &= \frac{-f(x)}{qx^q\Gamma(-q)} + \frac{qx^{1-q}f'(x)}{\Gamma(2-q)}
\end{aligned}\right\} \tag{14}$$

and

$$D^q f(x) \approx \frac{\Gamma(1+q)}{2\pi i}\sum_{k=1}^{4} R(\theta, n, L_k) + CRD\,, 0 < q < 1, \tag{15}$$

where

$$\left.\begin{aligned}
\theta(\zeta) &= \frac{f(\zeta)-G(\zeta)}{(\zeta-x)^{q+1}},\\
CRD = J_1 + J_2 + J_3 &= \frac{x^{-q}f(x)}{\Gamma(1-q)} + \frac{qx^{1-q}f'(x)}{\Gamma(2-q)} + \frac{q(q-1)x^{2-q}f''(x)}{2\Gamma(3-q)}
\end{aligned}\right\} \tag{16}$$

3. Numerical tests

The semi integral ($q = -1/2$), quarter integral ($q = -1/4$), semi derivative ($q = 1/2$) and quarter derivative ($q = 1/4$) have been found out in respect of the functions $e^x, sinx\ and\ 1/(1+x)$ for $x = 0.5$ by using the approximation formulas prescribed by equations (13)-(16). The fourteen point (n=14) transformed Gauss-Legendre rule has been employed for computing the differ integrals. Computed values have been appended in Table-I.

It is noted that $n = 14$ yields at least 15 decimal place accuracy. However, if the integrand in equation (7) is not modified and the factors CRI or CRD are dropped, the accuracy is only 8 decimal places at best.

Table-I

$f(x)$	q	Exact value of $D^q f(x)$	Approximation of $D^q f(x)$
e^x	-1/2	1.125564686969882	1.125564686969881
	-1/4	1.395620139949285	1.395620139949285
	1/2	1.923449247772747	1.923449247772747
	1/4	1.840872347142450	1.840872347142450
$sinx$	-1/2	0.258438983852738	0.258438983852738
	-1/4	0.358546311235634	0.358546311235634
	1/2	0.745530697780641	0.745530697780641
	1/4	0.613827443876775	0.613827443876775
$1/(1+x)$	-1/2	0.606668331316798	0.606668331316798
	-1/4	0.669734653640765	0.669734653640765
	1/2	0.329700263429644	0.329700263429644
	1/4	0.561907852438325	0.561907852438325

References:

[1] Dalir, M. and Bashour, M., 2010, "Applications of fractional calculus," Appl. Math.Sci., 4(21), pp. 1021-1032.

[2] Miller, K.S. Ross, B., 1993, An introduction to the fractional calculus and fractional differential equation, John Wiley & Sons, New York.

[3] Oldham, K.B., and Spanier, J., 2003, The fractional Calculus, Dover Publisher.

[4] Acharya, M., Mohapatra, S. N. and Acharya, B. P., 2011, "On numerical evaluation of fractional integrals," Appl. Math. Sci., 5(29), pp. 1401-1407.

[5] Diethelm, K. and Walz, G., 1997, "Numerical solution of fractional order differential equations by extrapolation," Numer. Algorith. 16, pp. 231-253.

[6] Lether, F. G., Cline, D. M. and Evans, O., 1985, "An error analysis for calculation of semiintegrals and semiderivatives by the RL algorithm," App. Math. Comp.," 17, pp. 45-67.

[7] Milovanovic, G. V., 2012, "Generalized quadrature formulae for analytic functions," Appl. Math. & computation, 218, pp. 8537-8551.

[8] Lether, F. G., 1976, "On Birkhoff-Young quadrature of analytic functions," J. Copm. Appl. Math., 2, pp. 81-84.

[9] Dannon, H. V., 2009, "The fundamental theorem of the fractional calculus and meaning of fractional derivatives," Gauge Inst. J., 5(1), pp. 1-26.