Expected Number of Real Zeros of Random Ultraspherical Polynomial

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Abstract

Let \( y_0 (w) , y_1 (w) , \ldots, y_n (w) \) be a sequence of independent random variables with mathematical expectation zero and variance one. We show that the polynomial \( \sum_{k=0}^{n} y_k (w) \psi_k (t) \), where \( \psi_k (t) \) is the normalized orthogonal ultraspherical polynomial, has \( n/\sqrt{3} \) real zeros for large values of \( n \).

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1. Introduction

Let \( f_n (t) = \sum_{k=0}^{n} b_k y_k (w) \phi_k (t) \) be a random polynomial, where \( y_0 (w) , y_1 (w) , \ldots, y_n (w) \) is a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance one; \( (\phi_0 (t) , \phi_1 (t) , \ldots) \) is a sequence of real valued polynomials (functions) and \( (b_0, b_1, \ldots) \) a sequence of real constants. Kac [4] showed that, when \( b_k=1 \) and \( \phi_k (t) = t^k \), the expected number of times the random polynomial \( f_n (t) \) crosses the \( t \)-axis is asymptotic to \( 1/\pi \log n \).

When \( b_0=0, b_k=1 \) for \( k \neq 0 \) and \( \phi_k (t) = \cos (\cos^{-1} t) \), J.E.A. Dunnage [2] estimated that the expected number of crossings of \( f_n (t) \) is asymptotic to \( 2n/\sqrt{3} \) in the interval \([-1, 1]\).

It is interesting to observe that while \( t^k \) s are a set of functions monotonic in \((-\infty, 0]\) and \([0, \infty) \), \( \cos (\cos^{-1} t) \) for each \( k \), oscillates \( k \) times between -1 and 1. The fact that the average number of zeros of \( y=0 \) when \( \phi_k (t) = \cos (\cos^{-1} t) \) is proportional
to the number of individual oscillations of \( \phi_k(t) \) about the t-axis, draws attention to the question as to how far the oscillatory nature of \( \phi_k(t) \) decisively affects the zeros of \( y = 0 \). Although the answer remains still inconclusive, we attempt to show that for large \( n \), the above equation may be expected to have \( c_0 > 0 \) number of real roots when \( \phi_k(t) \) happens to be the ultraspherical classical orthogonal Gebegauer polynomial. In other words the oscillatory property of \( \phi_k(t) \) is also shared by \( \sum_{k=0}^{n} y_k(w)b_k \phi_k(t) \). It was Das[1] who first initiated the work on Random Orthogonal polynomial. Taking \( b_k \phi_k(t) \) as normalized orthogonal Legendre Polynomial, he showed that expected number of real zeros of the polynomial is asymptotic to \( n/\sqrt{3} \).

Das’s polynomial can be considered as a special case of our polynomial \( f_n(t) \).

Now, \( \phi_k(t) \) is associated with a weight function \( u(t) = (1-t^2)^{-1/2} \) corresponding to the interval \((-1, 1)\) over which the integral of \( u(t) \phi_k^2(t) \) is a positive number \( h_k \). We take \( b_k = h_k^{1/2} \). Then the integral of \( \psi_k^2(t) = b_k^2 \phi_k^2(t) \) over the given interval is unity, so that each of the terms of the polynomial \( \sum_{k=0}^{n} y_k(w) \psi_k(t) \) \( = \sum_{k=0}^{n} y_k(w) b_k \phi_k(t) \) has same weightage in the same sense.

Thus, in what follows, we find the average number of zeros of the equation

\[
f_n(t) = \sum_{k=0}^{n} y_k(w) \psi_k(t) = 0. \tag{1.1}
\]

Let \( EN_n(a, b) \) denote the expected number of real zeros of \( f_n(t) \) in \((a, b)\). We prove the following theorem

**THEOREM:** If \( \{y_k(w)\}_{k=0}^{n} \) is a sequence of mutually independent, normally distributed random variables with mean zero and variance unity; Then

\[
EN_n(-\infty, \infty) \sim \frac{n}{\sqrt{3}}.
\]

**2. FORMULA FOR EN_n(a, b)**

Following the procedure of Kac [4], we obtain

\[
EN_n(f; a, b) = \frac{1}{\pi} \int_{a}^{b} \frac{[X_n(t)]^2 - [Y_n(t)]^2}{X_n(t)} \, dt \tag{2.1}
\]

where

\[
X_n(t) = X_n - \sum_{k=0}^{n} [\psi_k(t)]^2
\]
\[
Y_n(t) = Y_n - \sum_{k=0}^{n} [\psi_k(t)][\psi_k'(t)]
\]
\[
Z_n(t) = Z_n - \sum_{k=0}^{n} [\psi_k'(t)]^2
\]

provided that \( X_nZ_n - Y_n > 0 \) which holds good by Cauchy's inequality.

Let us put \( \mu = r_n^{-1} t^{-1} \) where \( r_n \) is the coefficient of \( t^n \) in \( \phi_n(t) \). The famous Christoffel-Darboux formula ([3], p.159) of the theory of orthogonal functions reads as follows:

\[
\sum_{k=0}^{n} h_k^{-1} \phi_k(\mu) \phi_k(t) = \mu_n \frac{[\phi_{n+1}(\mu)\phi_n(t) - \phi_n(\mu)\phi_{n+1}(t)]}{\mu - t} \tag{2.2}
\]
Putting \( \mu = t + \gamma \), expanding both the sides of (3.1.3) by Taylor’s series and equating the coefficients of \( \gamma \) we obtain that

\[
\sum_{k=0}^{n} \left[ \phi_k(t) \right] = \mu_n \left[ \phi_{n+1}(t) \right] \phi_n(t) - \phi_{n+1}(t) \phi'_n(t) ,
\]

(3.3)

Differentiating (2.4), we get

\[
\sum_{k=0}^{n} h_k^{-1} [ \phi_k(t) \phi''_n(t) ] = \frac{\mu_n}{2} [ \phi''_{n+1}(t) \phi_n(t) + \phi'_{n}(t) \phi''_{n+1}(t) - \phi'_n(t) ]
\]

(2.5)

Thus

\[
\text{EN}_n(f, a, b) = \frac{1}{\pi} b^b \ g_n(t) \ dt ,
\]

(2.6)

where

\[
\begin{align*}
g^2_n(t) &= \left[ \frac{W_n(t) + V_n(t)}{R_n(t)} - \frac{U_n(t)}{4R_n^2(t)} \right] , \\
R_n(t) &= \phi'_{n+1}(t) \phi_n(t) - \phi_{n+1}(t) \phi'_n(t) , \\
U_n(t) &= \phi''_{n+1}(t) \phi_n(t) - \phi'_{n+1}(t) \phi''_n(t) , \\
V_n(t) &= 1/2 [ \phi''_{n+1}(t) \phi'_n(t) - \phi''_n(t) ] \\
W_n(t) &= 1/6 [ \phi''_{n+1}(t) \phi'_n(t) - \phi''_n(t)]
\end{align*}
\]

3. Proof of the theorem

For the sake of convenience for proving the theorem, we break up the interval \((-1, 1)\) into three sub intervals namely (i) \((-1 + \epsilon, 1 + \epsilon)\) , (ii) \((-1, -1 + \epsilon)\) , and (iii) \((1 - \epsilon, 1)\) . We choose \( C = n^{-1/(4+\alpha)} \).

In the section 3.1, we find out the average number of zeros in the first interval. In the section 3.2 We prove that the number of zeros in the other two intervals are negligible in comparison to those in the interval (i).

EXPECTED NUMBER OF ZEROS IN THE INTERVAL \((-1 + \epsilon, 1 - \epsilon)\)

From [17, ], We have

\[
(1 - t^2) \phi''_n(t) = (2\lambda + 1) t \phi'_n(t) - n (n + 2\lambda) \phi_n(t)
\]

(3.1)

Thus

\[
\begin{align*}
2 (1 - t^2) V_n(t) &= n (n + 2\lambda) R_n(t) - (2n + 1 + 2\lambda) \phi_{n+1}(t) \phi'_n(t) , \\
(1 - t^2) U_n(t) &= (2\lambda + 1) t R_n(t) - (2n + 1 + 2\lambda) \phi_{n+1}(t) \phi_n(t)
\end{align*}
\]

(3.2)

(3.3)
\[ -2t\phi_n''(t) + (1-t^2)\phi_n'''(t) = (2\lambda+1)\phi_n(t) + (2\lambda+1) t\phi_n''(t) - n(n+2\lambda)\phi_n'(t). \]

Therefore it is easy to derive that

\[ 6\left(1-t^2\right)W_n(t) = \left(2\lambda+3\right)t \left[ \frac{(2\lambda+1)t}{(1-t^2)} R_n(t) - \frac{(2n+1+2\lambda)}{(1-t^2)} \phi_{n+1}(t) \phi_n(t) \right] + \left(\frac{2n+1+2\lambda}{(1-t^2)}\right) \phi_n(t) \left[ (n+1)t\phi_{n+1}(t) - \left(-(2\lambda+1)\right)\phi_n(t) \right]. \] (3.4)

From [17, p. 279], we have

\[ (1-t^2)\phi_{n+1}(t) = (2\lambda+n)\phi_n(t) - t\phi_{n+1}(t) \]

Therefore

\[ (1-t^2)R_n(t) = (\lambda+n)\phi_n(t) - t\phi_{n+1}(t) - (2\lambda+1+n)\phi_{n-1}(t)\phi_n(t). \] (3.5)

Gegenbauer polynomials can be considered as a special case of the Jacobi Polynomial \( P_n^{(\alpha,\beta)}(t) \), where \( \alpha = \beta = \lambda - \frac{1}{2} \) (see Szego[5] page 80). For large \( n \), we shall use the asymptotic estimate of \( \phi_n(t) \) as

\[ \phi_n(t) \sim \frac{2^{\lambda}}{(\pi n)^{1/2}} \left(1-t\right)^{-\frac{\lambda}{2}} \left(1+t\right)^{-\frac{\lambda}{2}} \left[ \cos\chi + o\left(\frac{1}{n\sin\theta}\right) \right], \] (3.6)

where \( \chi = (n\theta+\lambda\phi-\lambda\pi/2) \) and \( t=\cos \theta \). (We have taken \( \alpha = \beta = \lambda - \frac{1}{2} \) in asymptotic estimate of \( P_n^{(\alpha,\beta)} \) (Szego[5] page 190).

Hence from (3.5) and (3.6) we have

\[ R_n(t) = \frac{2^{2\lambda}}{n(1-t^2)} \left(1-t\right)^{-\lambda} \left(1+t\right)^{-\lambda} \left[ (1-t^2) + o\left(\frac{1}{n\sin\theta}\right) \right]. \]

It is easy to derive from (3.6) that

\[ \frac{\phi_{n+1}(t)\phi_n(t)}{R_n(t)} = o\left(\frac{1}{n\sin\theta}\right), \quad \frac{\phi_n(t)\phi_{n+1}(t)}{R_n(t)} = o\left(\frac{1}{n\sin\theta}\right) \]

and

\[ \frac{\phi_n(t)\phi_n(t)}{R_n(t)} = 0 \left(\frac{1}{(1-t^2)}\right). \]

Therefore, from (3.2) and (3.4) we have

\[ \frac{V_n(t)}{R_n(t)} \left(\frac{n(n+2\lambda)}{2(1-t^2)} - \frac{2(\lambda+1)}{1-t^2} \phi_{n+1}(t)\phi_n(t) \right) = \frac{n^2}{R_n(t)} \left[ 2(1-t^2) + o\left(\frac{n}{(1-t^2)^2}\right) \right], \]

\[ \frac{w_n(t)}{R_n(t)} \left[ \frac{(2\lambda+2)(2\lambda+1)^2}{6(1-t^2)} + o\left(\frac{n}{(1-t^2)^2}\right) \right] = \frac{n^2}{R_n(t)} \left[ 2(1-t^2) + o\left(\frac{n}{(1-t^2)^2}\right) \right]. \]

Thus,

\[ \frac{W_n(t)+V_n(t)}{R_n(t)} = 2\left(\frac{n^2}{(1-t^2)^2} + o\left(\frac{n}{(1-t^2)^2}\right) \right). \]

Also, we can derive from (3.3) that
\[
\frac{u_n(t)}{2R_n(t)} = 0 \left( \frac{1}{1-t^2} \right)^2 + O \left( \frac{1}{n(1-t^2)^2} \right).
\]

So
\[
g_n(t) = \frac{W_n(t) + u_n(t)}{R_n(t)} - \frac{u_n^2(t)}{4R_n^2(t)}
= n \sqrt{3(1-t^2)^{1/2}} \left[ 1 + O \left( \frac{1}{n(1-t^2)} \right) \right]^{1/2}
\]

For the range \((-1+\varepsilon, 1-\varepsilon)\), we notice that \(-1+t^2 > 2 \in \varepsilon^2 = \frac{2n^{(4+\delta)-1}}{n^{2(4+\delta)-1}}\), where \(\varepsilon = n^{(4+\delta)-1}\), as previously specified. Thus \((1-\varepsilon)^{-1} = 0(n^{(4+\delta)-1})\).

Therefore,
\[
g_n(t) = n \sqrt{3(1-t^2)} \left[ 1 + O \left( \frac{n^{2+\delta}}{4+\delta} \right) \right].
\]

Thus from (2.6)
\[
E_{N_n}(f; -1+\varepsilon, 1-\varepsilon)
= \int_{-1+\varepsilon}^{1-\varepsilon} \frac{n}{\sqrt{3(1-t^2)}} dt \left[ 1 + O \left( \frac{n^{2+\delta}}{4+\delta} \right) \right] \frac{n}{\sqrt{3}} \left[ 1 + O \left( \frac{n^{2+\delta}}{4+\delta} \right) \right]
\]

as \(\sin^{-1}(1-\varepsilon) \sim \pi/2\).

Number of zeros in \((-1, -1+\varepsilon)\) and \((1-\varepsilon, 1)\) the number of zeros of \((1, 1)\) is negligibly small in comparison to those in \((-1+\varepsilon, 1-\varepsilon)\).

Let
\[
f(z) = f(\tilde{y}(w), z) = \sum_{k=0}^{\infty} y_k(w)\psi_k(z),
\]

where \(\tilde{y}(w)\) denotes the random vector \((y_0(w), y_1(w), \ldots, y_n(w))\).

Now \(f(\tilde{y}(w), 1) = \sum_{k=0}^{\infty} y_k(w)\psi_k(1)\) is a random variable with mean zero and variance \(\sigma^2 = \sum_{k=0}^{\infty} \psi_k^2(1) \geq \psi_0^2(1) \geq 0\) and hence has the distribution function
\[
\frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} \exp \left( -\frac{v^2}{2\sigma^2} \right) dv.
\]

Now
\[
p \left( |f(1)| \leq e^{-2n\varepsilon} \right) = \left( \frac{2}{\pi\sigma^2} \right)^{1/2} \int_0^{e^{-2n\varepsilon}} \exp \left( -\frac{v^2}{2\sigma^2} \right) dv \leq \left( \frac{2}{\pi\sigma^2} \right)^{1/2} e^{-2n\varepsilon} < e^{-n\varepsilon}.
\]

(3.8)

Let \(I_n = \max_{0 \leq k \leq n} (|y_k(w)|)\). Then
\[
p(I_n \leq n) = p(\prod_{k=0}^{n} |y_k(w)| \leq n) = \prod_{k=0}^{n} (p(|y_k(w)| \leq n) = (\prod_{k=0}^{n} (1 - p(|y_k(w)| > n))
= \left[ 1 - \frac{\int_0^{\infty} e^{-v^2/2} dv}{\sqrt{\pi} n} \right]^{n+1} > 1 - e^{-n^2/2} (n > n_0).
\]

(3.9)
Let \( T_n = \max_{0 \leq k \leq n} |\psi_k (1 + 2e^{i\theta})| \). For the Gegenbauer polynomials, \( h_n = 2^{1-2\lambda n} \pi^{(n+\lambda)/(2\lambda+1)} \) for \( \lambda > 1/2 \) [see 17, p. 281]. Hence \( b_n = h_n^{-1/2} < \alpha_1 n^{-1/2} \) where \( \alpha_1 \) is a constant. From the integral representation of Gegenbauer polynomial [3], we have

\[
\phi_n(t) = \frac{2^{1-2\lambda n} \pi^{(2\lambda+n)/(2\lambda+1)}}{n! f(\lambda)^2} \int_0^\pi \left( t + i \sqrt{(1 - t^2)} \cos \theta \right)^n (\sin \theta)^{2\lambda-1} d\theta.
\]

Remembering that \( \eta = \alpha^{-1} \), we have from above representation

\[
\left| \phi^n (1 + 2 e^{i\theta}) \right| < \frac{2^{1-2\lambda n} (2\lambda+n)}{n! f(\lambda)^2} (1 + 2 e^n)^n < \alpha_3 n^{\alpha_2} (1 + 2 e^n)^n < \alpha_3 n^{\alpha_2} \exp \left[ 2n^{3+\delta} \right],
\]

where \( \alpha_2 \) and \( \alpha_3 \) are constants involving \( \lambda \) only.

Hence

\[
T_n = \phi_n(t) h_n^{-1/2} < An^{\alpha_2 + 1/2} \exp \left( \frac{3+\delta}{2n^{3+\delta}} \right),
\]

where \( A \) is a constant. Also

\[
|f (1 + 2 e^{i\theta})| = \left| \sum_{k=0}^{n} \gamma_k (w) \psi_k (1 + 2 e^{i\theta}) \right| \leq \sum_{k=0}^{n} |\gamma_k (w)| |\psi_k (1 + 2 e^{i\theta})| \leq 1, \quad \sum_{k=0}^{n} T_n = n t_n T_n.
\]

Hence from (3.9), it follows that \( P \left( |f (1 + 2 e^{i\theta})| \leq n^2 T_n \right) \geq 1 - e^{-n^2} \).

This together with (3.10), gives

\[
P \left( |f (1 + 2 e^{i\theta})| \leq An^{\alpha} \exp \left( 2n^{3+\delta} \right) \right) \geq 1 - e^{-n^2},
\]

where \( \alpha = \alpha_2 + 5/2 \).

So from (3.8) and (3.11), we obtain

\[
P \left( \frac{|f (1 + 2 e^{i\theta})|}{f (1)} \leq An^{\alpha} \exp \left( 2n^{3+\delta} + 2n \eta \right) \right) \geq 1 - e^{-n^2-n^{-n}e^1} > 1 - \frac{2}{n}. \tag{3.12}
\]

Let \( n (\eta) \) denote the number of zeros of \( f (\tilde{y}(w), z) = 0 \) inside the circle \(|z - 1| \leq \eta\). It is easy to see that the number of zeros of (2.1) inside the interval \( 1-\eta \leq t \leq 1 \) does not exceed \( n (\eta) \).

By Jensen's theorem, (Tetchmarsh[6]) we have

\[
n (\eta) \leq \frac{1}{2 n \log 2} \int_0^{2\pi} \log \left| f (1 + 2 e^{i\theta}) \right| f (1) d\theta \leq \frac{1}{2 n \log 2} \int_0^{2\pi} \log \left( An^{\alpha} \exp \left( 2n^{3+\delta} \right) + 2n \right) d\theta,
\]

for \( f (1) \neq 0 \), except for a set of measure at most \( 2/n \), as is evident from (3.12).

Thus, we obtain that the number of zeros of (1.1) in \((1-\epsilon, 1)\) is at most \( 0 \left( \frac{3+\delta}{n^{4+\delta}} \right) \), with probability at least \( 1 - \frac{2}{n} \). Identical result is obtainable for the number of zeros of (1.1) in \((-1, -1+\epsilon)\), so that
\[ EN_n(f; -1, -1+\varepsilon) = 0 \left( \frac{n^{3+\delta}}{n^{4+\delta}} \right). \]

The above derivation together with the estimate of \( EN_n(f; -1+\varepsilon, 1-\varepsilon) \), proves that
\[ EN_n(f; -1, 1) = \frac{n}{\sqrt{3}} + O \left( \frac{n^{3+\delta}}{n^{4+\delta}} \right). \]

Reference
