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A Note on Quai - umbilical Hypersurface of a Sasakian Manifold with $(\phi, g, u, v, \lambda)-$structure

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Abstract

In this paper we have studied the properties of Quasi – umbilical hypersurface M of a Sasakian manifold $\tilde{M}$ with $(\phi, g, u, v, \lambda)-$structure and established the relation for M to be cylindrical.

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1. Introduction

Let $\tilde{M}$ be a (2n + 1) - dimensional Sasakian manifold with a tensor field $\tilde{\phi}$ of type (1, 1), a fundamental vector field ξ and 1-form η such that

(1.1) $\eta(\xi) = 1$

(1.2) $\tilde{\phi}^2 = -I + \eta \otimes \xi$

where I denote the Identity transformation.

(1.3) (a) $\eta \circ \tilde{\phi} = 0$ (b) $\tilde{\phi}\xi = 0$ (c) rank ($\tilde{\phi}$) = 2n

If $\tilde{M}$ admits a Riemannian metric $\tilde{g}$, such that

(1.4) $\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)$

(1.5) $\tilde{g}(X, \xi) = \eta(X)$

then $\tilde{M}$ is said to admit a $(\tilde{\phi}, \xi, \eta, \tilde{g})$ - structure called Contact metric structure.
If moreover,

(1.6) $(\tilde{\nabla}_X \tilde{\phi})Y = \tilde{g}(X, Y)\xi - \eta(Y)X$

(1.7) $\tilde{\nabla}_X \xi = -\tilde{\phi}X$

where $\tilde{\nabla}$ denotes the Riemannian connection of the Riemannian metric $\tilde{g}$, then $(\tilde{M}, \tilde{\phi}, \xi, \eta, \tilde{g})$ is called a Sasakian manifold (see [9]). If we define $'F(X, Y) = \tilde{g}(\tilde{\phi}X, Y)$, then in addition to above relation we find

(1.8) $'F(X, Y) + 'F(Y, X) = 0$

(1.9) $'F(X, \tilde{\phi}Y) = 'F(Y, \tilde{\phi}X)$

(1.10) $'F(\tilde{\phi}X, \tilde{\phi}Y) = 'F(X, Y)$

2. Hypersurface of a Sasakian manifold with (ϕ, g, u, v, λ) - structure

Let us consider a 2n-dimensional manifold M embedded in $\tilde{M}$ with embedding $b : M \to \tilde{M}$. The map b induces a linear map B (called Jacobian map), $b : T_p \to T_{b_p}$.

Let an affine normal N of M is in such a way that $\tilde{\phi}N$ is always tangent to the hypersurface and satisfying the linear transformations.

(2.1) $\tilde{\phi}BX = B\phi X + u(X)N$

(2.2) $\tilde{\phi}N = -BU$

(2.3) $\xi = BV + \lambda N$

(2.4) $\eta(BX) = v(X)$

Where ϕ is a (1, 1) - type tensor; U, V are vector fields; u, v are 1-form and λ is a C^∞ functions. If $u \neq 0$ then, M is called a non-invariant hypersurface of $\tilde{M}$ (see [1] and [3]).

Operating (2.1), (2.2) and (2.3) and (2.4) by $\tilde{\phi}$ and using (1.1), (1.2) and (1.3) and taking tangent, normal parts separately, we get the following induced structure on M:

(2.5) **(a)** $\phi^2 X = -X + u(X)U + v(X)V$

(b) $u(\phi X) = \lambda v(X)$, $v(\phi X) = -\eta(N)u(X)$

(c) $\phi U = -\eta(N)V$, $\phi V = \lambda U$

(d) $u(U) = 1 - \lambda\eta(N)$, $u(V) = 0$

(e) $v(U) = 0$, $v(V) = 1 - \lambda\eta(N)$

and from (1.4) and (1.5) we get the induced metric g on M, i.e.,

(2.6) $g(\phi X, \phi Y) = g(X, Y) - u(X)u(Y) - v(X)v(Y)$

(2.7) $g(U, X) = u(X)$, $g(V, X) = v(X)$.

If we consider $\eta(N) = \lambda$, we get the following structures on M:

(2.8) **(a)** $\phi^2 = -I + u \otimes U + v \otimes V$

(b) $u \circ \phi = \lambda v$, $v \circ \phi = -\lambda u$

(c) $\phi U = -\lambda V$, $\phi V = \lambda U$

(d) $u(U) = 1 - \lambda^2$, $u(V) = 0$

(e) $v(U) = 0$, $v(V) = 1 - \lambda^2$

A manifold M with a metric g satisfying (2.6), (2.7) and (2.8) is called manifold with (ϕ, g, u, v, λ) - structure (see [2] and [7]). Let ∇ be the induced connection on the hypersurface M of the affine connection $\tilde{\nabla}$ of $\tilde{M}$.

Now using Gauss and Weingarten's equations

(2.9) $\tilde{\nabla}_{BX}BY = B\nabla_X Y + h(X,Y)N$

(2.10) $\tilde{\nabla}_{BX}N = BHX + \omega(X)N$

Where $g(HY,Z) = h(Y,Z)$, h and H are the second fundamental tensors of type (0, 2) and (1, 1), and ω is 1-form. Now differentiating (2.1), (2.2), (2.3) and (2.4) covariantly and using (2.9). (2.10), (1.6) and reusing (2.1), (2.2), (2.3) and (2.4), we get

(2.11) $(\nabla_Y \phi)(X) = v(X)Y - g(X,Y)V - h(X,Y)U - u(X)HY$

(2.12) $(\nabla_Y u)(X) = -h(\phi X,Y) - u(X)\omega(Y) - \lambda g(X,Y)$

(2.13) $(\nabla_Y v)(X) = g(X,\phi Y) + \lambda h(X,Y)$

(2.14) $\nabla_Y U = \omega(Y)U - \phi HY - \lambda Y$

(2.15) $\nabla_Y V = \phi Y + \lambda HY$

(2.16) $h(Y,V) = u(Y) - Y\lambda - \lambda\omega(Y)$

(2.17) $h(Y,U) = -u(HY)$

Since $h(X,Y) = g(HX,Y)$, then from (1.5) and (2.17), we get

(2.18) $h(Y,U) = 0 \Rightarrow HU = 0.$

3. Quasi – umbilical hypersurface: If

(3.1) $h(X,Y) = \alpha g(X,Y) + \beta q(X)q(Y)$

where α, β are scalar functions and q is 1-form such that $g(Q,X) = q(X)$, then M is called *Quasi – umbilical* hypersurface (see [9]). If $\alpha = 0$ and $\beta \neq 0$ then Quasi – umbilical hypersurface M is called *cylindrical* (see [9]). If $\alpha \neq 0$ and $\beta = 0$ then Quasi – umbilical hypersurface is called *totally umbilical*. If $\alpha = 0$ and $\beta = 0$ then Quasi – umbilical hypersurface is called *totally geodesic*.

Using (3.1) in (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17) we get

(3.2) $(\nabla_Y \phi)(X) = v(X)Y - g(X,Y)V - \{\alpha g(X,Y) + \beta q(X)q(Y)\}U - u(X)\{\alpha Y + \beta q(Y)Q\}$

(3.3) $(\nabla_Y u)(X) = -\{\alpha g(\phi X,Y) + \beta q(\phi X)q(Y)\} - u(X)\omega(Y) - \lambda g(X,Y)$

(3.4) $(\nabla_Y v)(X) = g(X,\phi Y) + \lambda\{\alpha g(X,Y) + \beta q(X)q(Y)\}$

(3.5) $\nabla_Y U = \omega(Y)U - \{\alpha\phi Y + \beta q(Y)Q\} - \lambda Y$

(3.6) $\nabla_Y V = \phi Y + \lambda\{\alpha Y + \beta q(Y)Q\}$

(3.7) $h(Y,V) = \alpha g(V,Y) + \beta q(Y)q(V)$

(3.8) $|u(Q)|^2 = -\dfrac{\alpha}{\beta}(1-\lambda^2)$

Also on a cylindrical hypersurface M with (ϕ,g,u,v,λ) - structure of a Sasakian manifold $\tilde{M}$, we have

(3.9) $v(\phi Q) = 0$

(3.10) $u(Q) = 0 \Leftrightarrow q(U) = 0$

Main Result

Now we prove the following:

3.1 Theorem

On the Quasi – umbilical hypersurface M with (ϕ,g,u,v,λ) - structure of a Sasakian manifold $\tilde{M}$, we have

(3.11) $q(V) = v(Q) = 0$

(3.12) $\omega(U) = \dfrac{1-\lambda^2}{\lambda} - \dfrac{U\lambda}{\lambda}$

(3.13) $\omega(U) = -\alpha\dfrac{(1-\lambda^2)}{\lambda} - \dfrac{V\lambda}{\lambda}$

Proof

Put X = U, Y = U in (3.1) and using (2.18), we have

$$\alpha(1-\lambda^2)+\beta\,|\,q(U)\,|^2=0 \Leftrightarrow |\,q(U)\,|^2=-\frac{\alpha}{\beta}(1-\lambda^2).$$

If we take X = V, Y = U in (3.1) then we get $\beta q(V)q(U)=0$. Since $\beta \neq 0$ and $q(U)\neq 0$, therefore $q(V)=v(Q)=0$ which proves (3.11).

Also from (3.1) and (2.16), we get

(3.14) $\alpha g(Y,V)+\beta q(Y)q(V)=u(Y)-Y\lambda-\lambda\omega(Y)\cdot$

Let Y = U and using $g(U,V)=u(V)=0$, we get (3.12). Further putting Y = V in (3.14) we have (3.13). □

If Quasi – umbilical hypersurface is cylindrical then from (3.13), we have $\omega=-d(\ell og\lambda)$. Therefore we can state the following theorem as a corollary of Theorem 3.1.:

3.1 Theorem

If Quasi – umbilical hypersurface M with (ϕ,g,u,v,λ) - structure of a Sasakian manifold $\tilde{M}$ is cylindrical, then 1-form ω satisfies the following relation

$$\omega = -d(\ell og\lambda).$$

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