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Some Random Fixed Point Theorems for Pair of Non Commuting Expansive type Mappings in Polish Space

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Abstract

The objective of this paper is to obtain some fixed point theorems for pair of non commuting expansive type multivalued operators on Polish space.

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1. Introduction

Random fixed point theorems are stochastic generalization of classical fixed point theorems [7, 15]. Itoh [9, 10] extended several well known fixed point theorems, i.e., for contraction, nonexpansive and condemning, mappings to the random case. Thereafter, various stochastic aspects of Schauder's fixed point theorem have been studied by Sehgal and Singh [16], Papageorgiou [14], Lin [12] and many authors. In a separable metric space, random fixed point theorems for contractive mappings were proved by Spacek [15], Hans [6 ,7, 8], Mukherjee [13]. Afterwards, Beg and Shahzad [2, 3], Badshah and Sayyed [4] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed points theorems for contraction random operators in Polish space.In present paper random fixed point theorems for pair of non commuting expansive type mapping in Polish space are investigated.

2. Preliminaries

Let (X, d) be a polish space, that is, a separable complete metric space and (Ω, A) be measurable space. Let 2^X be a family of all subsets of X and $CB(X)$ denote the family of all non-empty bounded closed subsets of X. A mapping $T : \Omega \to 2^X$ is called measurable if for all open subsets C of X, $T^{-1}(C) = \{\omega \in \Omega : T(\omega) \bigcap C \neq \phi\} \in A$. A mapping $\xi : \Omega \to X$ is said to be measurable selector of a measurable mapping $T : \Omega \to 2^X$ if ξ is measurable and $\xi(\omega) \in T(\omega)$ for all $\omega \in \Omega$. A mapping $f : \Omega \times X \to X$ is called a random operator if for all $x \in X$, $f(;x)$ is measurable. A mapping $T : \Omega \times X \to CB(X)$ is called a random multivalued operator, if for every $x \in X$, $T(;x)$ is measurable. A measurable mapping $T : \Omega \times X$ is called random fixed point of a random multivalued operator $T : \Omega \times X \to CB(X)$ $(f : \Omega \times X \to X)$, if for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))(f(\omega, \xi(\omega)) = \xi(\omega))$. Let $T : \Omega \times X \to CB(X)$ be a random operator and $\{\xi_n\}$ a sequence of measurable mappings $\xi_n : \Omega \to X$.The sequence $\{\xi_n\}$ is said to be asymptotically T-regular if $d\big(\xi_n(\omega), T(\omega, \xi_n(\omega))\big) \to 0$.

3. Main Results

Theorem 3.1. Let X be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting continuous surjective random multivalued operators. If there exist measurable mappings $\alpha, \beta, \gamma : \Omega \to (0, 1)$ such that

$$\begin{aligned} &H(ST(\omega, x), TS(\omega, y)) \\ &\quad \geq \frac{\alpha(\omega)d(x, ST(\omega, x))d(x, y) + \beta(\omega)d(y, TS(\omega, y))d(x, y) + \gamma(\omega)d(x, ST(\omega, x))d(y, TS(\omega, y))}{d(x, ST(\omega, x)) + d(y, TS(\omega, y))} \end{aligned} \tag{3.1.1}$$

for each $x, y \in X, \omega \in \Omega$ and $d(x, ST(\omega, x)) + d(y, TS(\omega, y)) \neq 0$ where $\alpha, \beta, \gamma \in R^+$, with $\alpha(\omega) > 0$, $\beta(\omega) > 0$, $\alpha(\omega) + \beta(\omega) + \gamma(\omega) > 2$. Then ST and TS have a common fixed point.

(Here H represents the Hausdroff metric on $CB(X)$ induced by the metric d).

Proof. We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2$............

$$\begin{aligned} \xi_{2n}(\omega) &= ST(\omega, \xi_{2n+1}(\omega)) \\ \xi_{2n+1}(\omega) &= TS(\omega, \xi_{2n+2}(\omega)) \end{aligned} \tag{3.1.2}$$

Now consider

$$d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) = H\big[ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))\big]$$

$$\geq \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))}$$
$$+ \frac{\beta(\omega)d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))}$$
$$+ \frac{\gamma(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))}$$

$$= \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) + \beta(\omega)d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))}$$
$$+ \frac{\gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))}{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))}$$

$$\Rightarrow d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))\big[d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))\big]$$
$$\geq \alpha(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) + \beta(\omega)d^2(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$
$$+ \gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))$$

$$\Rightarrow d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) + d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))$$
$$\geq d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))(\alpha(\omega) + \beta(\omega) + \gamma(\omega))$$
$$\times \min\big\{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\big\}$$

$$\Rightarrow d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))(\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 1)$$
$$\min\big\{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\big\}$$

Case I

$$d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq \big[\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 1\big]d^2(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$
$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq \Big(\frac{1}{\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 1}\Big)^{1/2} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$
$$\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k_1(\omega)\ d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

where

$$k_1 = k_1(\omega) = \Big(\frac{1}{\alpha(\omega) + \beta(\omega) + \gamma(\omega) - 1}\Big)^{1/2} < 1\ [As\ \alpha(\omega) + \beta(\omega) + \gamma(\omega) > 2]$$

Similarly we can calculate

$$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k_1(\omega)\ d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

where

$$k_1 = k_1(\omega) = \big(\frac{1}{\alpha(\omega)+\beta(\omega)+\gamma(\omega)-1}\big)^{1/2} < 1 \; [As\ \alpha(\omega)+\beta(\omega)+\gamma(\omega) > 2]$$

and so on

Case II

$$\begin{aligned} d^2(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) &\geq \big[\alpha(\omega)+\beta(\omega)+\gamma(\omega)-1\big] \\ &\quad \times d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \\ \Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) &\leq \frac{1}{\alpha(\omega)+\beta(\omega)+\gamma(\omega)-1}\, d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \\ \Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) &\leq k_2(\omega)\, d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \end{aligned}$$

where

$$k_2 = k_2(\omega) = \frac{1}{\alpha(\omega)+\beta(\omega)+\gamma(\omega)-1} < 1 \; [As\ \alpha(\omega)+\beta(\omega)+\gamma(\omega) > 2]$$

Similarly we can calculate

$$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k_2(\omega)\, d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))$$

where

$$k_2 = k_2(\omega) = \big(\frac{1}{\alpha(\omega)+\beta(\omega)+\gamma(\omega)-1}\big) < 1 \; [As\alpha(\omega)+\beta(\omega)+\gamma(\omega) > 2]$$

and so on

So, in general

$$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k\ d(\xi_{n-1}(\omega), \xi_n(\omega)) \text{ for } n = 1, 2, 3.....$$

where $k = k(\omega) = max\{k_1(\omega), k_2(\omega)\} < 1$

$$\Rightarrow d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k^n \;\; d(\xi_0(\omega), \xi_1(\omega))$$

Now we shall prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. For this for every positive integer p we have,

$$\begin{aligned} d(\xi_n(\omega), \xi_{n+p}(\omega)) &\leq d(\xi_n(\omega), \xi_{n+1}(\omega)) + d(\xi_{n+1}(\omega), \xi_{n+2}(\omega)) + \cdots \\ &\quad + d(\xi_{n+p-1}(\omega), \xi_{n+p}(\omega)) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+p-1}) d(\xi_0(\omega), \xi_1(\omega)) \\ &= k^n(1 + k + k^2 + \cdots + k^{p-1})\, d(\xi_0(\omega), \xi_1(\omega)) \\ &< \frac{k^n}{(1-k)} d(\xi_0(\omega), \xi_1(\omega)) \end{aligned}$$

which tends to zero as $n \to \infty$. It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that

$$\xi_n(\omega) \to \xi(\omega) \;\text{ for each }\; \omega \in \Omega. \tag{3.1.3}$$

Existence of random fixed point

Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions $g : \Omega \to X$ and $g^{'} : \Omega \to X$ such that

$$\xi(\omega) \in ST(\omega, g(\omega)) \text{ and } \xi(\omega) \in TS(\omega, g^{'}(\omega)) \tag{3.1.4}$$

Consider

$$d(\xi_{2n}(\omega), \xi(\omega)) = H\big(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g^{'}(\omega))\big)$$

$$\geq \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(\xi_{2n+1}(\omega), g^{'}(\omega)) + \beta(\omega)d(g^{'}(\omega), TS(\omega, g^{'}(\omega)))d(\xi_{2n+1}(\omega), g^{'}(\omega))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g^{'}(\omega), TS(\omega, g^{'}(\omega)))}$$
$$+ \frac{\gamma(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega)))d(g^{'}(\omega), TS(\omega, g^{'}(\omega)))}{d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + d(g^{'}(\omega), TS(\omega, g^{'}(\omega)))}$$

$$= \frac{\alpha(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(\xi_{2n+1}(\omega), g^{'}(\omega)) + \beta(\omega)d(g^{'}(\omega), \xi(\omega)))d(\xi_{2n+1}(\omega), g^{'}(\omega))}{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(g^{'}(\omega), \xi(\omega))}$$
$$+ \frac{\gamma(\omega)d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))d(g^{'}(\omega), \xi(\omega))}{d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + d(g^{'}(\omega), \xi(\omega))}$$

As $\{\xi_{2n}(\omega)\}$ and $\{\xi_{2n+1}(\omega)\}$ are subsequences of $\{\xi_n(\omega)\}$ as $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$, $\{\xi_{2n+1}(\omega)\} \to \xi(\omega)$ (using 3.1.3).

Therefore

$$d(\xi(\omega), \xi(\omega)) \geq \frac{\alpha(\omega)d(\xi(\omega), \xi(\omega))d(\xi(\omega), g^{'}(\omega)) + \beta(\omega)d(g^{'}(\omega), \xi(\omega))d(\xi(\omega), g^{'}(\omega))}{d(\xi(\omega), \xi(\omega)) + d(g^{'}(\omega), \xi(\omega))}$$
$$+ \frac{\gamma(\omega)d(\xi(\omega), \xi(\omega))d(g^{'}(\omega), \xi(\omega))}{d(\xi(\omega), \xi(\omega)) + d(g^{'}(\omega), \xi(\omega))}$$

$$\Rightarrow \beta(\omega)d(g^{'}(\omega), \xi(\omega)) \leq 0$$
$$\Rightarrow d(g^{'}(\omega), \xi(\omega)) = 0 \ [As \ \ \beta(\omega) > 0]$$

$$\Rightarrow g^{'}(\omega) = \xi(\omega) \tag{3.1.5}$$

In an exactly similar way (using $\alpha(\omega) > 0$) we can prove that

$$\Rightarrow \xi(\omega) = g(\omega) \tag{3.1.6}$$

The fact (3.1.4) along with (3.1.5) and (3.1.6) show that $\xi(\omega)$ is a common fixed point of ST and TS. This completes the proof of the theorem 3.1. ■

Theorem 3.2. Let X be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting continuous surjective random multivalued operators. If there exist measurable mappings $\alpha_1, \alpha_2, \alpha_3 : \Omega \to (0, 1)$ such that

$$H(ST(\omega, x), TS(\omega, y)) \geq \alpha_1(\omega)d(x, ST(\omega, x)) + \alpha_2(\omega)d(y, TS(\omega, y)) + \alpha_3(\omega)d(x, y) \tag{3.1.7}$$

for each $x, y \in X$, $\omega \in \Omega$ $\alpha_1, \alpha_2, \alpha_3 \in R^+$ and $\alpha_1(\omega) + \alpha_2(\omega) + \alpha_3(\omega) > 1$ and $\alpha_3(\omega) > 0$. Then ST and TS have a common fixed point.

(Here H represents the Hausdroff metric on $CB(X)$ induced by the metric d)

Proof. We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2, \ldots$

$$\begin{aligned} \xi_{2n}(\omega) &= ST(\omega, \xi_{2n+1}(\omega)) \\ \xi_{2n+1}(\omega) &= TS(\omega, \xi_{2n+2}(\omega)) \end{aligned} \tag{3.1.8}$$

Now we put $x = \xi_{2n+1}(\omega)$ and $y = \xi_{2n+2}(\omega)$ in (3.1.7) we get

$$\begin{aligned} H(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))) \geq\ & \alpha_1(\omega)d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) \\ &+ \alpha_2(\omega)d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega))) \\ &+ \alpha_3(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \end{aligned}$$

$$\begin{aligned} = \alpha_1(\omega)&d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + \alpha_2(\omega)d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)) \\ &+ \alpha_3(\omega)d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \end{aligned}$$

$$\begin{aligned} &\Rightarrow (1 - \alpha_1(\omega))d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq (\alpha_2(\omega) + \alpha_3(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \\ &\Rightarrow (\alpha_2(\omega) + \alpha_3(\omega))d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq (1 - \alpha_1(\omega))d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \end{aligned}$$

$$\begin{aligned} &\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq \frac{1 - \alpha_1(\omega)}{\alpha_2(\omega) + \alpha_3(\omega)} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \\ &\Rightarrow d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k_1(\omega)d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \end{aligned}$$

where $k_1 = k_1(\omega) = \dfrac{1 - \alpha_1(\omega)}{\alpha_2(\omega) + \alpha_3(\omega)} < 1$ *[As $\alpha_1(\omega) + \alpha_2(\omega) + \alpha_3(\omega) > 1$].*

Similarly we can calculate

$$\Rightarrow d(\xi_{2n+2}(\omega), \xi_{2n+3}(\omega)) \leq k_1(\omega)\ d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$$

where

$$k_1 = k_1(\omega) = \frac{1-\alpha_1(\omega)}{\alpha_2(\omega)+\alpha_3(\omega)} < 1 \; [As \; \alpha_1(\omega)+\alpha_2(\omega)+\alpha_3(\omega) > 1]$$

and so on.

So, in general

$$\begin{aligned} & d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k \; d(\xi_{n-1}(\omega), \xi_n(\omega)) \;\text{ for }\; n = 1, 2, 3, \ldots \\ \Rightarrow \; & d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq k^n \;\; d(\xi_0(\omega), \xi_1(\omega)) \end{aligned} \tag{3.1.9}$$

Now we can prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. (As proved in theorem 3.1) and since X is complete, so there exist a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each

$$\omega \in \Omega \tag{3.1.10}$$

Existence of random fixed point:

Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions $g : \Omega \to X$ and $g^{'} : \Omega \to X$ such that

$$\xi(\omega) \in ST(\omega, g(\omega)) \;\text{ and }\; \xi(\omega) \in TS(\omega, g^{'}(\omega)) \tag{3.1.11}$$

Consider

$$d(\xi_{2n}(\omega), \xi(\omega)) = H\big(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g^{'}(\omega))\big)$$

$$\begin{aligned} & \geq \alpha_1(\omega) d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))) + \alpha_2(\omega) d(g^{'}(\omega), TS(\omega, g^{'}(\omega))) \\ & \quad + \alpha_3(\omega) d(\xi_{2n+1}(\omega), g^{'}(\omega)) \\ & = \alpha_1(\omega) d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)) + \alpha_2(\omega) d(g^{'}(\omega), \xi(\omega)) + \alpha_3(\omega) d(\xi_{2n+1}(\omega), g^{'}(\omega)) \end{aligned}$$

As $\{\xi_{2n}(\omega)\}$ and $\{\xi_{2n+1}(\omega)\}$ are subsequences of $\{\xi_n(\omega)\}$ as $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$, $\{\xi_{2n+1}(\omega)\} \to \xi(\omega)$ (using 3.1.10).

Therefore

$$d(\xi(\omega), \xi(\omega)) \geq \alpha_1(\omega) d(\xi(\omega), \xi(\omega)) + \alpha_2(\omega) d(g^{'}(\omega), \xi(\omega)) + \alpha_3(\omega) d(\xi(\omega), g^{'}(\omega))$$

$$\begin{aligned} & 0 \geq [\alpha_2(\omega) + \alpha_3(\omega)] d(\xi(\omega), g^{'}(\omega)) \\ & \quad \Rightarrow d(\xi(\omega), g^{'}(\omega)) = 0 \;\; [\alpha_2(\omega) + \alpha_3(\omega) > 0] \\ & \quad \Rightarrow \xi(\omega) = g^{'}(\omega) \end{aligned} \tag{3.1.12}$$

In an exactly similar way (using $\alpha_1(\omega) + \alpha_3(\omega) > 0$) we can prove that

$$\Rightarrow \xi(\omega) = g(\omega) \tag{3.1.13}$$

The fact (3.1.11) along with (3.1.12) and (3.1.13) show that $\xi(\omega)$ is a common fixed point of ST and TS. This completes the proof of the theorem 3.2. ■

Theorem 3.3. Let X be a polish space. Let $S, T : \Omega \times X \to CB(X)$ be two non commuting continuous multivalued operators. If there exists measurable mapping $\alpha : \Omega \to (0, 1)$ such that

$$H(ST(\omega, x), TS(\omega, y)) \geq \alpha(\omega) \min \left\{d(x, y), d(x, ST(\omega, x)), d(y, TS(\omega, y))\right\} \tag{3.1.14}$$

for each $x, y \in X$, $\omega \in \Omega$, and $\alpha(\omega) > 1$. Then ST and TS have a common fixed point.

Proof. We define a sequence $\{\xi_n(\omega)\}$ for each $\omega \in \Omega$ as follows for $n = 0, 1, 2, \ldots$

$$\begin{aligned} \xi_{2n}(\omega) &= ST(\omega, \xi_{2n+1}(\omega)) \\ \xi_{2n+1}(\omega) &= TS(\omega, \xi_{2n+2}(\omega)) \end{aligned} \tag{3.1.15}$$

Now we put $x = \xi_{2n+1}(\omega)$ and $y = \xi_{2n+2}(\omega)$ in (3.1.14) we get

$$\begin{aligned} H(ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, \xi_{2n+2}(\omega))) \geq \alpha(\omega) \min \big\{&d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)), \\ &d(\xi_{2n+1}(\omega), ST(\omega, \xi_{2n+1}(\omega))), \\ &d(\xi_{2n+2}(\omega), TS(\omega, \xi_{2n+2}(\omega)))\big\} \end{aligned}$$

$$= \alpha(\omega) \min \left\{d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)), d(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega))\right\}$$
$$\Rightarrow d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq \alpha(\omega) \min \left\{d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))\right\}$$

Case I

$$d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \geq \alpha(\omega) d(\xi_{2n+1}(\omega), \xi_{2n}(\omega))$$

$\Rightarrow 1 \geq \alpha(\omega)$ which is contradiction.

Case II

$$d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq \frac{1}{\alpha(\omega)} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$
$$d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \leq k(\omega) d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$$

where

$$k = k(\omega) = \frac{1}{\alpha(\omega)} < 1$$

[As $\alpha(\omega) > 1$]. So, in general

$$\begin{aligned} d(\xi_n(\omega), \xi_{n+1}(\omega)) &\leq k\ d(\xi_{n-1}(\omega), \xi_n(\omega)) \text{ for } n = 1, 2, 3, \ldots \\ \Rightarrow d(\xi_n(\omega), \xi_{n+1}(\omega)) &\leq k^n\ \ d(\xi_0(\omega), \xi_1(\omega)) \end{aligned} \tag{3.1.16}$$

Now we can prove that for each $\omega \in \Omega$ $\{\xi_n(\omega)\}$ is a Cauchy sequence. (As proved in theorem 3.1) and since X is a complete space so there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$.

Existence of random fixed point:

Since S and T are surjective maps so ST and TS are also surjective and hence there exist two functions $g : \Omega \to X$ and $g^{'} : \Omega \to X$ such that

$$\xi(\omega) \in ST(\omega, g(\omega)) \text{ and } \xi(\omega) \in TS(\omega, g^{'}(\omega)) \tag{3.1.17}$$

Consider

$$\begin{aligned} d(\xi_{2n}(\omega), \xi(\omega)) = H\big(&ST(\omega, \xi_{2n+1}(\omega)), TS(\omega, g^{'}(\omega))\big) \\ &\geq \alpha(\omega) \min \big\{d(\xi_{2n+1}(\omega), g^{'}(\omega)), d(\xi_{2n+1}(\omega), \\ &ST(\omega, \xi_{2n+1}(\omega))), d(g^{'}(\omega), TS(\omega, g^{'}(\omega)))\big\} \end{aligned}$$

$$= \alpha(\omega) \min \big\{d(\xi_{2n+1}(\omega), g^{'}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n}(\omega)), d(g^{'}(\omega), \xi(\omega))\big\}$$

As $\{\xi_{2n}(\omega)\}$ and $\{\xi_{2n+1}(\omega)\}$ are subsequences of $\{\xi_n(\omega)\}$ as $n \to \infty$, $\{\xi_{2n}(\omega)\} \to \xi(\omega)$, $\{\xi_{2n+1}(\omega)\} \to \xi(\omega)$.

Therefore

$$\begin{aligned} d(\xi(\omega), \xi(\omega)) &\geq \alpha(\omega) \min \big\{d(\xi(\omega), g^{'}(\omega)), d(\xi(\omega), \xi(\omega)), d(g^{'}(\omega), \xi(\omega))\big\} \\ 0 &\geq \alpha(\omega) d(\xi(\omega), g^{'}(\omega)) \\ &\Rightarrow d(\xi(\omega), g^{'}(\omega)) = 0\ \ [As\ \ \alpha(\omega) > 1] \\ &\Rightarrow \xi(\omega) = g^{'}(\omega) \end{aligned} \tag{3.1.18}$$

In an exactly similar way we can prove that

$$\Rightarrow \xi(\omega) = g(\omega) \tag{3.1.19}$$

The fact (3.1.17) along with (3.1.18) and (3.1.19) show that $\xi(\omega)$ is a common fixed point of ST and TS. This completes the proof of the theorem 3.3. ■

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