A Common Fixed Point Theorem for Six Self Mappings under $D^*$–compatibility

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Abstract

The object of this paper is to introduce the concept of $D^*$-compatibility of pair of self maps in a metric space. Using this concept we try to establish a unique common fixed point theorem for six self mappings which satisfy a more general contractive inequality of integral type. The result obtained is a significant generalization of result of general contractive condition of integral type reported earlier and the result obtained is novel.

Keywords: compatible maps, metric space, common fixed point.

Introduction

There are numerous generalizations of the Banach contraction principle. After an interesting result of Kannan [5] many existence theorems dealing with the mappings satisfying various types of contractive conditions. In 1972 Bianchini [1] established a new result by using different contractive condition [5]. Branciari [2] obtained a fixed point theorem for a mapping for an integral type inequality. Rhoades [6] proved two fixed point theorem involving more general contraction conditions. Many authors proved common fixed point theorem using the concept of weakly compatible mapping [4]. The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [3] as a generalization of commuting mappings. Sahu and Sharma [8] introduced the notion of $D$-compatibility in a set $X$ which is not
necessarily metric space. Sahu and Dewangan [7] introduced the notation of $D^*$-compatibility as a generalization of $D$-compatibility in d-topological spaces and established some fixed point theorems. It is important to note that every $D$-compatible pair is $D^*$-compatible but converse is not true. [Example 1] One can easily observe that the notion of $D^*$-compatibility is an important generalization of various known commuting and non-commuting mappings. The work reported here is the generalization of P. Vijayaraju, B.E. Rhoades and R. Mohanraj [9]. A fixed point theorems for a map satisfying a general contractive condition of Integral type.

**Preliminaries**

**Definition**

Let $A$ and $S$ be two mappings from a metric space $(X, d)$ into itself. The mappings $A$ and $S$ are said to be compatible if $d(Ax_n, Sx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Sx_n = Ax_n = x$, for some $x \in X$.

**Definition**

Let $X$ be a non empty set and $A, S: X \to X$ be two self mappings. Then $\{A, S\}$ is said to be $D$–compatible if $Au = Su$ for some $u \in X \Rightarrow ASu = SAu$.

**Definition**

The self mappings $A$ and $S$ defined on a metric space $(X, d)$ is said to be $D^*$–compatible if $Au = Su$ for some $u \in X \Rightarrow d(A^2u, Au) = d(S^2u, Su)$. The following example shows that $D^*$-compatibility is more general than $D$-compatibility.

**Example 1.** Let $X = [0, 2]$ with usual metric. Let $A, S: X \to X$ be a mappings defined as

- $Ax = 2 - x$, if $0 \leq x \leq 1$
- $Sx = x + 1$, if $0 \leq x \leq 1$
- $= 0$, if $1 \leq x \leq 2$
- $= 2$, if $1 \leq x \leq 2$

Then $x = \frac{1}{2}$ is a coincidence point of $A$ and $S$, So $d(S^{\frac{1}{2}}x, S^{\frac{1}{2}}x) = d(A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) = 1$.

Then $\{A, S\}$ is $D^*$-compatible. But $AS^{\frac{1}{2}} \neq SA^{\frac{1}{2}}$. So $\{A, S\}$ is not $D$-compatible.

**Proposition 1**

Every $D$-compatible pair is $D^*$-compatible.

**Proof**

Let $(X, d)$ be a metric space and $A, S: X \to X$ be two self mappings. Let the pair $\{A, S\}$ be compatible. Then $Au = Su$ for some $u \in X \Rightarrow ASu = SAu$. Hence

\[ d(A^2u, Au) = d(ASu, Au) \]

\[ = d(SAu, Su) \]

\[ = d(S^2u, Su). \]

Therefore, $\{A, S\}$ is $D^*$-compatible.

Example above shows that the converse of proposition is not true in general.
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Result

Theorem.

Let $A$, $B$, $S$, $T$, $P$ and $Q$ be six self mappings from metric space $(X, d)$ into itself satisfying the following conditions:

- $S(X) \subset PQ(X)$, $T(X) \subset AB(X)$;
- Pair $(S, AB)$ and $(T, PQ)$ are $D^*$-compatible;
- Either $AB$ or $PQ$ is complete subspace of $X$;
- $AB = BA$, $PQ = QP$, $SB = BS$, $TQ = QT$;
- $\int_0^d (Sx, Ty) \varphi(t) dt \leq \psi(\int_0^d (ABx, PQy) \varphi(t) dt)$ for all $x, y \in X$, where
- $M(ABx, PQy) = \max \{d(ABx, PQy), d(Sx, ABx), d(Ty, PQy), \frac{d(Sx, PQy) + d(Ty, ABx)}{2}\}$, where $\varphi \in \Phi$, $\psi \in \Psi$.

Then $A$, $B$, $S$, $T$, $P$ and $Q$ have a unique common fixed point in $X$.

Proof

Let $x_0$ be any point in $X$. From (3.1.1) there exists $x_1, x_2 \in X$ such that

- $Sx_0 = PQx_1$, and $Tx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

- $y_{2n} = Sx_{2n} = PQx_{2n+1}$
- $y_{2n+1} = Tx_{2n+1} = ABx_{2n+2}$.

For $n = 1, 2, 3, \ldots$.

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

- $\int_0^d (Sx_{2n}, Tx_{2n+1}) \varphi(t) dt = \int_0^d (y_{2n}, y_{2n+1}) \varphi(t) dt$

$\leq \psi(\int_0^d (ABx_{2n}, PQx_{2n+1}) \varphi(t) dt)$

$\leq \psi(\int_0^d (y_{2n-1}, y_{2n}) \varphi(t) dt)$

Put $x = x_{2n+1}$ and $y = x_{2n+2}$ in (3.1.5), we have

- $\int_0^d (Sx_{2n+1}, Tx_{2n+2}) \varphi(t) dt = \int_0^d (y_{2n+1}, y_{2n+2}) \varphi(t) dt$
\[ \leq \psi \left( \int_0^{M(ABX_{2n+1}, PQX_{2n+2})} \varphi(t) \, dt \right) \]
\[ \leq \psi \left( \int_0^{d(y_{2n}y_{2n+1})} \varphi(t) \, dt \right). \]

Set \( d_0 = \int_0^{d(y_ny_1)} \varphi(t) \, dt \). Therefore for each \( n \geq 0 \),
\[ \int_0^{d(y_ny_{n+1})} \varphi(t) \, dt \leq \psi \left( \int_0^{d(y_{n-1}y_n)} \varphi(t) \, dt \right) \leq \psi^2(d_0). \]

Let \( m, n \in \mathbb{N} \). Using Triangular inequality, we get
\[ d(y_n, y_m) \leq \sum_{i=1}^{m-1} d(y_i, y_{i+1}). \]

It can be shown by induction that
\[ \int_0^{d(y_ny_m)} \varphi(t) \, dt \leq \sum_{i=1}^{m-1} \int_0^{d(y_{i+1}y_{i+1})} \varphi(t) \, dt. \]

Then
\[ \int_0^{d(y_ny_m)} \varphi(t) \, dt \leq \sum_{i=1}^{m-1} \psi^i(d_0) \leq \sum_{i=1}^{\infty} \psi^i(d_0). \]

Taking limit as \( n, m \to \infty \) and using condition for (3.1.4), it follows that \( \{y_n\} \) is a Cauchy sequence in \( X \). Suppose that \( AB(X) \) is complete. So \( \{y_{2n}\} \) converges to a point \( z \in X \), i.e., \( z=ABu \) for some \( u \in X \). Also its subsequence converges to the same point \( z \), i.e.,
\[ \{SX_{2n}\} \to z \quad \text{and} \quad \{TX_{2n+1}\} \to z \]
\[ \{PQX_{2n+1}\} \to z \quad \text{and} \quad \{ABX_{2n+2}\} \to z. \]

Put \( x=u \) and \( y=x_{2n+1} \) in (3.1.5), we have
\[ \int_0^{d(Su, Tx_{2n+1})} \varphi(t) \, dt \leq \psi \left( \int_0^{M(ABu, PQX_{2n+1})} \varphi(t) \, dt \right). \]

Where \( M(ABu, PQX_{2n+1}) = \max \{ d(ABu, PQX_{2n+1}), d(Su, ABu) \} \).

\[ \frac{d(Su, PQX_{2n+1}) + d(Tx_{2n+1}ABu)}{2}, \]

Taking limit as \( n, m \to \infty \), we have
\[ M(ABu, z) = \max \{ d(z, z), d(Su, z), d(z, z), \frac{d(Su, z) + d(x, z)}{2} \} \]
\[ = \max \{ 0, d(Su, z), 0, \frac{d(Su, z) + d(x, z)}{2} \} = \max \{ d(Su, z), \frac{d(Su, z)}{2} \}. \]

Thus,
\[ \int_0^{d(Su, z)} \varphi(t) \, dt \leq \psi \left( \int_0^{d(Su, z)} \varphi(t) \, dt \right). \]
Which implies that Su=z. So, Su=ABu=z.

Since S (X) ⊂ PQ (X), there exists a point v ∈ X such that Su=PQv=z.

Put x=u and y=v in (3.1.5), we have
\[ \int_0^1 \varphi(t) \, dt \leq \psi \left( \int_0^1 M(ABu, PQv) \varphi(t) \, dt \right) \]

Where \( M(ABu, PQv) = \max \left\{ d(ABu, PQv), \frac{d(z, PQv) + d(Tz, ABu)}{2} \right\} \),

Taking limit as \( n, m \to \infty \), we have
\[ M(z, PQv) = \max \left\{ d(z, PQv), \frac{d(z, PQv) + d(Tz, ABu)}{2} \right\} \]

\( \leq d(Tz, z) \).

Thus, \( \int_0^1 \varphi(t) \, dt \leq \psi \left( \int_0^1 \varphi(t) \, dt \right) \) for all x, y \in X.

Which implies that Tz=z. So, Tz=PQz=z.

Since \( \{T, PQ\} \) is \( D^* \)-compatible and Tz=PQz=z, it follow that
\( d(Tz, z) = d(TTv, Tz) = d(PQ^2v, PQz) = d(PQz, z) \).

Put x=u and y=v in (3.1.5), we have
\[ \int_0^1 \varphi(t) \, dt \leq \psi \left( \int_0^1 M(ABu, PQz) \varphi(t) \, dt \right) \]

Where \( M(ABu, PQz) = \max \left\{ d(ABu, PQz), \frac{d(z, PQz) + d(Tz, ABu)}{2} \right\} \),

\( = \max \left\{ d(z, PQz), \frac{d(z, PQz) + d(Tz, ABu)}{2} \right\} \),

\( = \max \left\{ d(z, PQz), \frac{d(z, PQz) + d(Tz, ABu)}{2} \right\} \),

\( = d(PQz, z) \).

So, \( \int_0^1 \varphi(t) \, dt = \psi \left( \int_0^1 M(ABu, PQz) \varphi(t) \, dt \right) \)

Which implies that PQz=z and Tz=z.
Next, Since $S_u = ABu = z$, then From $D^*$-compatibility of \{S, AB\}, we have
\[
d(S_z, z) = d(S^2u, Su) = d(PQ^2u, PQz) = d(PQz, z)
\]
Put $x = z$ and $y = v$ in (3.1.5), we have
\[
\int_0^d (S_z, Tz) \varphi(t) dt \leq \psi \left( \int_0^M (ABz, PQv) \varphi(t) dt \right) \quad \text{for all } x, y \in X.
\]
Where \( M(ABz, PQv) = \max\{d(ABz, PQv), d(Sz, ABz), d(Tv, PQv), \frac{d(Sz, PQv) + d(Tv, ABz)}{2} \} \),
\[
M(ABz, PQv) = \max\{d(ABz, z), d(Sz, z), d(z, z), \frac{d(Sz, z) + d(z, ABz)}{2} \},
\]
\[
= \max\{d(ABz, z), d(Sz, z) + d(ABz, z), \frac{d(Sz, z) + d(z, ABz)}{2} \},
\]
\[
= d(ABz, z)
\]
So, \[
\int_0^d (S_z, Tz) \varphi(t) dt = \psi \left( \int_0^M (ABz, PQv) \varphi(t) dt \right) \leq \psi \left( \int_0^d (ABz, z) \varphi(t) dt \right).
\]
Which implies that $ABz = z$ and $Sz = z$.
Hence $PQz = ABz = Sz = Tz = z$.

Put $x = z$ and $y = Qz$ in (3.1.5), we have
\[
\int_0^d (S_z, TQz) \varphi(t) dt \leq \psi \left( \int_0^M (ABz, PQQz) \varphi(t) dt \right) \quad \text{for all } x, y \in X.
\]
Where \( M(ABz, PQQz) = \max\{d(ABz, PQQz), d(Sz, ABz), d(TQz, PQQz), \frac{d(Sz, PQQz) + d(TQz, ABz)}{2} \} \),
\[
\text{Since } PQ = QP, TQ = QT
\]
PQ (Qz) = QP (Qz) = Q (PQz) = Q
TQz = QTz = Qz.
\[
M(ABz, PQQz) = \max\{d(z, Qz), d(z, z), d(Qz, Qz), \frac{d(z, Qz) + d(Qz, z)}{2} \},
\]
\[
= \max\{d(z, Qz), 0, 0, \frac{d(Sz, z) + d(Qz, z)}{2} \},
\]
\[
\leq d(Qz, z).
\]
So, \[
\int_0^d (Qz, z) \varphi(t) dt = \psi \left( \int_0^M (ABz, PQQz) \varphi(t) dt \right) \leq \psi \left( \int_0^d (Qz, z) \varphi(t) dt \right).
\]
Which implies that $Qz = z$
Also we have $PQz = z$, hence $Pz = z$.
Therefore $PQz = Pz = Qz = z$. 
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Now Put $x = Bz$ and $y = z$ in (3.1.5), we have
\[ \int_0^1 d(SBz, Tz) \varphi(t) \, dt \leq \psi \left( \int_0^1 M(ABBz, PQz) \varphi(t) \, dt \right) \text{ for all } x, y \in X. \]

Where $M(ABBz, PQz) = \max \{ d(ABBz, PQz), d(SBz, ABBz), d(Tz, PQz), \frac{d(SBz, PQz) + d(Tz, ABBz)}{2} \}$,

Since $AB = BA$, $SB = BS$

$ABz = BAz = BA(Bz) = B(ABz) = Bz$

$SBz = BSz = Bz$.

$M(ABBz, PQz) = \max \{ d(Bz, z), d(Bz, Bz), d(z, z), \frac{d(ABz, PQz) + d(z, z)}{2} \}$,

\[ \leq d(Bz, z). \]

So, \[ \int_0^1 d(Bz, z) \varphi(t) \, dt = \psi \left( \int_0^1 M(ABBz, PQz) \varphi(t) \, dt \right) \leq \psi \left( \int_0^1 d(Bz, z) \varphi(t) \, dt \right). \]

Which implies that $Bz = z$

Also we have $ABz = z$, hence $Az = z$.

Therefore $ABz = Az = Bz = z$.

Thus we have $Az = Bz = Sz = Pz = Qz = Tz = z$.

i.e., $z$, is common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$.

**Uniqueness**

Let $w$ be another common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$.

Then $Aw = Bw = Pw = Sw = Tw = Qw = w$.

Put $x = z$ and $y = w$ in (3.1.5) we have
\[ \int_0^1 d(Sz, Tw) \varphi(t) \, dt \leq \psi \left( \int_0^1 M(ABz, PQw) \varphi(t) \, dt \right) \text{ for all } x, y \in X. \]

Where $M(ABz, PQw) = \max \{ d(ABz, PQw), d(Sz, ABz), d(Tw, PQw), \frac{d(Sz, PQw) + d(Tw, ABBz)}{2} \}$,

$M(ABz, PQw) = \max \{ d(z, w), d(z, z), d(w, w), \frac{d(z, w) + d(z, z)}{2} \} = d(z, w)$

So, \[ \int_0^1 d(Sz, Tw) \varphi(t) \, dt = \psi \left( \int_0^1 M(ABz, PQw) \varphi(t) \, dt \right) \leq \psi \left( \int_0^1 d(z, w) \varphi(t) \, dt \right). \]
Which implies that $z=w$. Therefore, $z$ is the unique common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$.

In the same manner it can be show that $z$ is the unique common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$ when $PQ$ is complete.

References


