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Observations on the Non-homogenous Biquadratic Equation with Four Unknowns

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Abstract

We obtain infinitey many non-zero integer quadruples (x,y,z,w) satisfying the biquadratic equation with four unknowns $8(x^3+y^3)=(1+3k^2)z^3w$.. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers are obtained.

Keywords: bi-quadratic equation with four unknowns,integral solution, special numbers.

MSC Subject Classification: 11D25

Notations

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2))}{2}\right]$$

$$P_n^m=\left[\frac{n(n+1))}{6}\right][(m-2)n+(5-m)]$$

$$OH_n=\frac{1}{3}n(2n^2+1)$$

$$SO_n=n(2n^2-1)$$

$$S_n=6n(n-1)+1$$

$$PR_n=n(n+1)$$

$$J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$$

$$j_n=2^n+(-1)^n$$

$$PT_n=\frac{n(n+1)(n+2)(n+3)}{4}$$

Introduction

The biquadratic diophntine(homogeneous or non-homogeneous) equation offer an unlimited field for research due to their variety[1-3]. In particular, one may refer [4-15] for ternary non-honogenous biquadratic equations. This communication concerns with yet another interesting ternary non-homogenous biquadratic equation given by $8(x^3+y^3)=(1+3k^2)z^3w$. A few interesting relations between special polygonal numbers, pyramidal numbers and special number patterns are exhibited.

Method of Analysis

The non-homogeneous biquadratic Diophantine equation with four unknowns to be solved for getting non-zero integral solutions is

$$8(x^3+y^3)=(1+3k^2)z^3w \quad (1)$$

To start with, the substitution of the linear transformations

$$x=u+v, y=u-v, w=16u \quad (2)$$

in (1), leads to

$$u^2+3v^2=(1+3k^2)^n z^3 \quad (3)$$

The above equation (3) is solved through three different patterns and thus, one can obtain three distinct sets of solutions to (1).

Pattern 1

Let $z=a^2+3b^2$ (4)

Taking n=0 in (3), we have

$$u^2+3v^2=z^3 \quad (5)$$

whose solution is given by

$$u_0=a^3-9ab^2$$

$$v_0=3a^2b-3b^3$$

Again taking n=1 in (3), we have

$$u^2+3v^2=(1+3k^2)z^3 \quad (6)$$

whose solution is represented by

$$u_1=u_0-3kv_0$$

$$v_1=ku_0+v_0$$

The general form of integral solutions to (3) is given by

$$\begin{pmatrix}u_s\\v_s\end{pmatrix}=\begin{pmatrix}A_s & i\sqrt{3}B_s\\ -\frac{i}{\sqrt{3}}B_3 & A_s\end{pmatrix}\begin{pmatrix}u_0\\v_0\end{pmatrix}, \text{ s=1,2,3,…}$$

where

$$A_s = \frac{\left(1+ik\sqrt{3}\right)^s + \left(1-ik\sqrt{3}\right)^s}{2}$$

$$B_s = \frac{\left(1+ik\sqrt{3}\right)^s - \left(1-ik\sqrt{3}\right)^s}{2}$$

Thus in view of (2), the following quadruple (x_s, y_s, z, w_s) of integers based on (x_0, y_0, z_0, w_0) also satisfy (1)

$$x_s(a,b) = (u_0+v_0)A_s + i\sqrt{3}\left(v_0 - \frac{u_0}{3}\right)B_s$$

$$y_s(a,b) = (u_0+v_0)A_s + i\sqrt{3}\left(v_0 + \frac{u_0}{3}\right)B_s$$

$$z(a,b) = a^2 + 3b^2 \quad w_s(a,b) = 16(u_0 A_s + i\sqrt{3}v_0 B_s)$$

The above values of x_s, y_s, w_s satisfy the following recurrence relations respectively.

$$x_{s+2} - 2x_{s+1} + (3k^2+1)x_s = 0$$

$$y_{s+2} - 2y_{s+1} + (3k^2+1)y_s = 0$$

$$w_{s+2} - 2w_{s+1} + (3k^2+1)w_s = 0$$

Properties

1) $a[x_s(a,1) + y_s(a,1)] = 8A_s\left(6PT_{a-1} - 3P^3{}_{a-1} - 2t_{4,a}\right) + 36i\sqrt{3}B_s$
2) $3a[x_s(a,1) - y_s(a,1)] = 108P^3{}_{a-1}A_s - 2i\sqrt{3}B_s\left(24PT_{a-1} - 12P^3{}_{a-1} - 8t_{4,a}\right)$
3) $6x_s(a,1) = 3A_s\left(4P^5{}_a + 3S_a - 14t_{4,a} - 9\right) - 2i\sqrt{3}B_s\left(6P^3{}_{a-1} - 9PR_a + 2t_{3,a} - t_{4,a} + 9\right)$
4) $w_s(2^{2\alpha}+1) = 2^4\left(A_s\left(3J_{6\alpha} - 9j_{2\alpha} + 10\right) + 3\sqrt{3}iB_s\left(j_{4\alpha} - 2\right)\right)$
5) $w_s(a+1,1) = 16\left(A_s\left(6P^3{}_{a-1} + 2t_{5,a} - 2Gno_a - 10\right) + \sqrt{3}iB_s\left(3PR_a + 5t_{4,a} - 2t_{7,a}\right)\right)$

Pattern 2

Substituting (4) in (3) and using the method of factorization, define

$$\left(u + i\sqrt{3}v\right) = \left(1+ik\sqrt{3}\right)^n\left(a + i\sqrt{3}b\right)^3$$

Expanding binomially and equating real and imaginary parts, we have

$$u = f(k)(a^3 - 9ab^2) - 9g(k)(a^2b - b^3)$$

$$v = g(k)(a^3 - 9ab^2) + 3f(k)(a^2b - b^3)$$

where

$$\left.\begin{aligned} f(k) &= \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r nC_{2r} k^{2r} 3^r \\ g(k) &= \sum_{r=1}^{\left[\frac{n+1}{2}\right]} (-1)^{r-1} nC_{2r-1} k^{2r-1} 3^{r-1} \end{aligned}\right\} \quad (7)$$

In view of (2) and (7) the corresponding integer solution (x,y,z,w) to (1) is obtained as

$$x = (f(k) + g(k))(a^3 - 9ab^2) + (3f(k) - 9g(k))(a^2 b - b^3)$$
$$y = (f(k) + g(k))(a^3 - 9ab^2) - (3f(k) - 9g(k))(a^2 b - b^3)$$
$$z = a^2 + 3b^2$$
$$w = 16[f(k)(a^3 - 9ab^2) - 9g(k)(a^2 b - b^3)]$$

Pattern 3

Substituting (4) in (3) and using the method of factorization, define

$$u + i\sqrt{3}v = \left(1 + ik\sqrt{3}\right)^n \left(a + i\sqrt{3}b\right)^3$$
$$= r^n \exp(in\theta)\left(a + i\sqrt{3}b\right)^3$$

where $r = \sqrt{3k^2 + 1}$, $\theta = \tan^{-1} k\sqrt{3}$ (8)

Equating real and imaginary parts in (8), we get

$$u = r^n\left(\left(a^3 - 9ab^2\right)\cos n\theta - \left(3a^2 b - 3b^3\right)\sqrt{3}\sin n\theta\right)$$
$$v = r^n\left(\left(a^3 - 9ab^2\right)\frac{\sin n\theta}{\sqrt{3}} - \left(3a^2 b - 3b^3\right)\cos n\theta\right)$$

In view of (2) and (4), the corresponding values of x,y,z and w are represented by

$$x(a,b) = r^n\left(\left(a^3 - 9ab^2 + 3a^2 b - 3b^3\right)\cos n\theta + \left(a^3 - 9ab^2 - 9a^2 b + 9b^3\right)\frac{\sin n\theta}{\sqrt{3}}\right)$$
$$y(a,b) = r^n\left(\left(a^3 - 9ab^2 - 3a^2 b + 3b^3\right)\cos n\theta + \left(-a^3 + 9ab^2 - 9a^2 b + 9b^3\right)\frac{\sin n\theta}{\sqrt{3}}\right)$$
$$z(a,b) = a^2 + 3b^2$$
$$w(a,b) = r^n\left(\left(a^3 - 9ab^2\right)\cos n\theta - \left(3a^2 b - 3b^3\right)\sqrt{3}\sin n\theta\right)$$

Properties

1) $x(a,a) = 4\left(1 + 3k^2\right)^{\frac{n}{2}}\left(\frac{\sin n\theta}{\sqrt{3}} + \cos n\theta\right)\left(SO_a + 2t_{3,a} - t_{4,a}\right)$

2) $8x(a,b)+8y(a,b)-w(a,b)=0$
3) $3\left(z\left(2^n,1\right)-j_{2n}\right)$ is a nasty number.
4) $2\left(z\left(2^n,1\right)-3J_{2n}\right)$ is a cubical integer.

Conclusion
One may search for other patterns of solutions and their corresponding properties.

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