Partially Balanced Incomplete Block Designs Arising from Minimum Total Dominating Sets in a Graphs

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Abstract

In this paper we determine the number of minimum total dominating sets of paths and cycles and prove that the set of all minimum total dominating sets of a cycle forms a partially balanced incomplete block design. We also determine all cubic graphs on ten vertices in which the set of all minimum total dominating sets forms a Partially Balanced Incomplete Block Design.

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Introduction

The relation between Graph theory and partially balanced incomplete block designs (PBIBD) is not a new one and R. C. Bose, in his pioneering paper [1], established the relation between PBIBDs and strongly regular graphs. R. C. Bose[2], has shown that strongly regular graphs emerge from PBIBD; with 2 - association schemes. Harary et.al., [3][4], considered the relation between isomorphic factorization of regular graphs and PBIBD with 2 - association scheme. Ionin and M.S. Shrikhande [5] studied certain kind of designs called (v, k, λ , μ) - designs over strongly regular
graphs. Walikar et al., introduced another kind of design called (v, β₀, μ) - designs whose blocks are maximum independent sets in regular graphs on v vertices.

In this paper, we establish the link between PBIBD and graphs through the collection of minimum total dominating sets. We prove that set of all minimum total dominating sets of cycle forms a PBIBD. We also determine all cubic graphs on ten vertices in which the set of all minimum total dominating sets forms a PBIBD.

Throughout this paper, G = (V, E) stands for a finite, connected, undirected graph with neither loops nor multiple edges. Terms not defined here are used in the sense of Harary [6]. Cₙ and Pₙ are cycle and path on n vertices respectively. A set D is a total dominating set if for every vertex v ∈ V there exists vertex u ∈ D, u ≠ v such that u is adjacent to v or A subset D of V is called total dominating set in G, if the induced subgraph (D) has no isolated vertices. The minimum cardinality of a total dominating set of G is a total domination number of G denoted by γₜ(G) or γₜ. A total dominating set of cardinality γₜ is called minimum total dominating set or γₜ- set. The fundamentals of total domination in graphs and several advanced topics in domination are given in Haynes et al., [7] [8].

Definitions and Preliminary Results

Definition
[2] Given a set {1, 2, 3, ..., v}, a relation satisfying the following conditions is said to be an association scheme with m-classes.

(i) Any two symbols α, β are i-th associates for some i, with 1 ≤ i ≤ m and this relation of being i-th associates is symmetric.
(ii) The number of i-th associates of each symbol is nᵢ.
(iii) If α and β are two symbols which are i-th associates, then the number of symbols which are j-th associates of α and k-th associates of β is p_{jk}’s and is independent of the pair of i-th associates α and β.

Definition
[2] Consider the set of symbols V = {1, 2, 3, ..., v} and association scheme with m classes. A partially balanced incomplete block design (PBIBD) is a collection of b subsets of S, each of cardinality k (k < v), such that every symbol occurs in exactly r subsets and two symbols α, β which are i-th associates occur together in λᵢ sets, the number λᵢ being independent of choice of pair α, β. The numbers v, b, r, k, λᵢ (i = 1, 2, 3, ..., m) are called the parameters of the first kind and the numbers nᵢ’s and p_{jk}’s of first definition are called the parameters of second kind.

Theorem
[9] For any integer n ≥ 1, γₜ(Pₙ) = ⌈n/2⌉ + ⌈n/4⌉ - ⌈n/4⌉

Theorem
[9] For any integer n ≥ 1, γₜ(Cₙ) = ⌈n/4G(G)⌉
Minimum Total Dominating Sets in Paths

Let $M^0_{\gamma_t}(G)$ denote the number of minimum total dominating sets in $G$. In this section we find $\gamma_t(P_n)$ for $n = 4k$, $4k + 1$, $4k + 2$, $4k + 3$ for any $k \geq 7$. Where $\gamma_t$ is total domination number.

**Theorem**

$$M^0_{\gamma_t}(P_n) = \begin{cases} 
1 & \text{if } n = 4k, \ k \geq 1 \\
2 & \text{if } n = 4k + 1, \ k \geq 7 \\
5 & \text{if } n = 4k + 2, \ k \geq 7 \\
2 & \text{if } n = 4k + 3, \ k \geq 7 
\end{cases}$$

**Proof:** Let $n = (v_1, v_2, v_3, \ldots, v_n)$ be the path on $n$-vertices

**Case 1:** $n = 4k, \ k \geq 1$

Then $\gamma_t(P_n) = 2k$ and $\{v_i, v_{i+1} / 1 \equiv 2 mod(4)\}$, $1 \leq i \leq 4k$ is a unique $\gamma_t$-set of $P_n$

Therefore $M^0_{\gamma_t}(G) = 1$

**Case 2:** $n = 4k+1, \ k \geq 7$

Then $\gamma_t(P_n) = 2k + 1$

There exists exactly two $\gamma_t$ sets of $P_n$ containing $v_1$ and $v_{4k}$

Clearly $D_1 = \{v_i, v_{i+1} \& v_{4k} / i \equiv 2 mod(4)\}$

And $D_2 = \{v_2, v_1, v_{i+1} / i \equiv 3 mod(4)\}$, $1 \leq i \leq 4k$ are the $\gamma_t$-sets of $P_n$

To prove these are the only two sets

Consider $D = D_1 - \{v_2, v_4k\}$ or $D = D_2 - \{v_2, v_4k\}$ is not a $\gamma_t$-set of $P_n$

Here $v_3$ and $v_{4k}$ become isolate

Which contradicts the definition of total dominating set.

Thus there are only two $\gamma_t$-sets for $P_{4k+1}$

Therefore $M^0_{\gamma_t}(P_n) = 2$

**Case 3:** $n = 4k+2, \ k \geq 7$

Then $\gamma_t(P_n) = 2k + 2$

There exists exactly five $\gamma_t$-sets of $P_n$, which are as follows,

$D_1 = \{v_1, v_2, v_4, v_5, \ldots, v_{4k}, v_{4k+1}\}$,

$D_2 = \{v_2, v_3, v_5, v_7, \ldots, v_{4k+1}, v_{4k+2}\}$,

$D_3 = \{v_2, v_3, v_5, v_7, \ldots, v_{4k}, v_{4k+1}\}$,

$D_4 = \{v_2, v_3, v_4, v_7, v_8, v_11, v_12, \ldots, v_{4k}, v_{4k+1}\}$,

$D_5 = \{v_2, v_3, v_4, v_5, v_9, v_12, v_{13}, \ldots, v_{4k}, v_{4k+1}\}$

Therefore $M^0_{\gamma_t}(P_n) = 5$

**Case 4:** $n = 4k + 3$, then $\gamma_t(P_n) = 2k + 2$

There exists exactly two $\gamma_t$-sets containing $v_2$, $v_{4k+3}$ and $v_3, v_{4k+2}$

That is $D_i = \{v_j, v_{j+1} / j \equiv i mod(4)\}$, $1 \leq i \leq 2$ and $1 \leq j \leq 4k+2$ are the $\gamma_t$-set of $P_n$

Therefore $M^0_{\gamma_t}(P_n) = 2$
Minimum Total Dominating Sets in Cycles

Let $M^0_{\gamma_t}(G)$ denote the number of minimum total dominating sets in $G$ and $\gamma_t(G)$ is total domination number.

In this section we find the values of $\gamma_t(C_n)$ for $n = 4k, 4k+1, 4k+2, 4k+3$.

We have $\gamma_t(C_n) = \left\lceil \frac{n}{\Delta(G)} \right\rceil$, where $n = 4k$

We proceed to determine $M^0_{\gamma_t}(C_n)$ for cycles.

Theorem

$M^0_{\gamma_t}(C_n) = \begin{cases} 4 & \text{if } n = 4k, k \geq 2 \\ 4k+1 & \text{if } n = 4k + 1, k \geq 2 \\ 4k+2 & \text{if } n = 4k + 2, k \geq 2 \\ 4k+3 & \text{if } n = 4k + 3, k \geq 2 \end{cases}$

Proof. Let $C_n = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n, v_1\}$ be a cycle on $n$ vertices

Case 1: $n = 4k$, $k \geq 2$

$\gamma_t(C_{4k}) = \left\lceil \frac{4k}{2} \right\rceil = 2k$

Then $D_i = \{v_j, v_{j+1} / j \equiv i \mod(4)\}$, $1 \leq i \leq 4k$, $j = 1, 2, 3, \ldots, 4k$ generates $\gamma_t$-sets.

Therefore $M^0_{\gamma_t}(C_{4k}) = 4$

Case 2: $n = 4k+1$, $\gamma_t(C_{4k+1}) = 2k + 1$, $k \geq 2$

$D_i = \{v_i, v_{i+1}, v_{i+1} / j \equiv i \mod(4)\}$, $1 \leq i \leq 4k + 1$, $j = 1, 2, 3, \ldots, 4k+1$ generates $\gamma_t$-sets.

Therefore $M^0_{\gamma_t}(C_{4k+1}) = 4k + 1$

Case 3: $n = 4k+2$, $\gamma_t(C_{4k+2}) = 2k + 2$, $k \geq 2$

$D_i = \{v_j, v_{j+1} / j \equiv i \mod(4)\}$, $1 \leq i \leq 4k + 2$ generates $\gamma_t$-sets.

Therefore $M^0_{\gamma_t}(C_{4k+2}) = 4k + 2$

Case 4: $n = 4k+3$, $\gamma_t(C_{4k+3}) = 2k + 2$, $k \geq 2$

$D_i = \{v_j, v_{j+1} / j \equiv i \mod(4)\}$, $1 \leq i \leq 4k + 3$, $j = i, \ldots, 4k+3$ generates $\gamma_t$-sets.

Therefore $M^0_{\gamma_t}(C_{4k+3}) = 4k + 3$

Minimum Total Dominating Sets & PBIBDs

We now proceed to establish a relation between the set of all minimum total dominating sets and PBIBDs for cycles and some cubic graphs on ten vertices.

Definition

A graph $G$ is called PBIB graph if the set of all minimum total dominating sets of $G$ forms a PBIBD with a suitable $m$-association scheme.

Theorem

The collection of all minimum total dominating sets of a cycle $C_n$, where $n = 4k, k \geq 2$ are the blocks of PBIBD with $3$-association scheme and parameters $v = n$, $b = 4$, $r = 2$, $k = 4\lambda_i$, $i = 1, 2, 3$. 
**Proof.** Let $C_n = \{v_1, v_2, v_3, \ldots, v_n, v_1\}$

By the theorem 4.1 $M^0_{\gamma_t}(C_{4k}) = 4$ and four $\gamma_t$- sets are given by

\[ D_i = \{v_j, v_{j+1} / j \equiv i \mod (4)\}, \ 1 \leq i \leq 4k, j = 1, 2, 3, \ldots, 4k. \]

Two distinct vertices $u$ and $v$ are said to be first associates if $d(u, v) \equiv 0 \mod (4)$; second associates if $d(u, v) \equiv 0 \mod (2)$, third associates if $d(u, v) = 0$.

Clearly the parameters of second kind are given by

\[ n_1 = \frac{n}{4}, n_2 = \frac{n}{2} \text{ and } n_3 = \frac{n}{4} - 1 \text{ and} \]

\[ p^1 = \begin{pmatrix} 0 & 0 & \frac{n}{4} - 1 \\ 0 & \frac{n}{2} & 0 \\ \frac{n}{4} - 1 & 0 & 0 \end{pmatrix}, \quad p^2 = \begin{pmatrix} 0 & \frac{n}{4} & 0 \\ \frac{n}{4} & 0 & \frac{n}{4} - 1 \\ 0 & \frac{n}{4} - 1 & 0 \end{pmatrix}, \quad p^3 = \begin{pmatrix} \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{2} & 0 \\ 0 & 0 & \frac{n}{4} - 2 \end{pmatrix} \]

The four $\gamma_t$ - sets $D_1, D_2, D_3$ and $D_4$ are the blocks of PBIBD with parameters $v = n, b = 4, r = 2, k = 4, \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

**Remark**

If $G$ is a PBIB graph then the number of $\gamma_t$ – sets containing any particular vertex $v$, is the same for all vertices in $G$.

We now proceed to determine all the cubic graphs with ten vertices in which the set of all minimum total dominating sets forms a PBIBD. We observe that if $G$ is a cubic graph on ten vertices then $\gamma_t(G) = 4$. There are 21 cubic graphs on ten vertices which are given below.

![Figure 1: G₁ to G₉](image-url)
We prove that $G_{18}$ is the only cubic graph which forms a PBIBD.

**Theorem**
The graphs $G_i = 1 \leq i \leq 21$, $i \neq 18$ are not PBIB graphs.

**Proof.** We prove the theorem for the graphs $G_1, G_2$ and $G_{19}$ and proofs are similar for the remaining cases.
In $G_1$, there are exactly two $\gamma$-sets containing 1 and there are three $\gamma$-sets containing 4 and hence $G_1$ is not a PBIB graph.

In $G_2$, there are exactly two $\gamma$-sets containing 1 and there are three $\gamma$-sets containing 3.

In $G_{19}$, there are exactly eight $\gamma$-sets containing 1 and there are seven $\gamma$-sets containing 3.

Thus each vertex is not appearing in a fixed number of blocks; hence these will not form PBIBD.

**Theorem**
The only graph which forms a PBIBD is $G_{18}$

Proof. There exists a parameters of first kind as $(10, 10, 4, 4, 2, 1)$ and parameters of second kind as

$$p^1 = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$$ and $$p^2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$n_1 = 3, \lambda_1 = 2$ and $n_2 = 6, \lambda_2 = 1$

Which proves that only connected cubic graph on ten vertices that is $G_{18}$ forms a PBIBD.

**Conclusion & Scope**
In this paper we consider $(v, b, r, k)$-design over cycles and enumerating the minimum total dominating sets of cycles and paths. One may also consider the designs whose block are subsets of vertex set of regular graph with given property such as vertex cover, edge independent sets and many other properties associated with edge set and vertex set of graph. Exploration of designs of this sort may provide considerable insight in to the construction of designs and even may lead to the construction of some special types of codes in coding theory.

**References**


