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Coefficent estimates for bi-univalent strongly starlike and Bazilevic functions

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Abstract

We consider meromorphic univalent strongly starlike and strongly Bazilevic functions that are bi-univalent and we find coefficient estimates for these types of functions. A function is said to be strongly starlike bi-univalent or strongly Bazilevic bi-univalent if both the function and its inverse are strongly starlike univalent or strongly Bazilevic univalent.

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1. Introduction

A function is said to be univalent on some open domain if the images of distinct points in that domain are distinct. We let $\mathcal{S}$ denote the class of univalent functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ defined on the open unit disk $\mathbb{D} := \{z : |z| < 1\}$.

The well-known Koebe one-quarter theorem asserts that the function $f \in \mathcal{S}$ has an inverse defined on the disk $\mathbb{D}_\rho := \{z : |z| < \rho\}$, $(\rho \geq \frac{1}{4})$. Thus, the inverse of $f \in \mathcal{S}$ is a univalent analytic function on the disk $\mathbb{D}_\rho$. The function $f \in \mathcal{S}$ is called *bi-univalent* in $\mathbb{D}$ if f^{-1} is also univalent in $\mathbb{D}$. The class σ of bi-univalent analytic functions was introduced by Lewin [11] and it was shown that $|a_2| < 1.51$. Brannan and Clunie [3] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Later, Netanyahu [12] proved that $\max_{f\in\sigma} |a_2| = 4/3$. Brannan and Taha [4] and Taha [17] also investigated certain subclasses of bi-univalent analytic functions and found estimates for their initial coefficients. Recently, Ali *et al.* [1], Frasin and Aouf [6] and Srivastava *et al.* [16] found estimates for coefficients a_2 and a_3 of certain subclasses of bi-univalent functions.

Let Σ denote the class of meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n},$$

defined on the domain $\Delta := \{z : 1 < |z| < \infty\}$. Since $g \in \Sigma$ is univalent, it has an inverse g^{-1} that satisfy

$$g^{-1}(g(z)) = z \quad (z \in \Delta),$$

and

$$g(g^{-1}(w)) = w \quad (M < |w| < \infty,\ M > 0)\,.$$

The inverse function $h = g^{-1}$ has a series expansion of the form

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n},$$

in some neighborhood of infinity, say $M < |w| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \Sigma$ is said to be *meromorphic bi-univalent* if $h = g^{-1} \in \Sigma$.

Estimates on the coefficients of meromorphic univalent functions were investigated in the literature. For example, Schiffer [13] obtained the estimate $|b_2| \leq 2/3$ for $g \in \Sigma$ if $b_0 = 0$. Duren [5] gave an elementary proof of the inequality $|b_n| \leq 2/(n+1)$ for $g \in \Sigma$ if $b_k = 0$ for $1 \leq k < n/2$. The case for the inverse of meromorphic univalent functions is not as easy as it may sound. Springer [15] proved $|B_3| \leq 1$ and $|B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$,and conjectured that $|B_{2n-1}| \leq [(2n-2)!]/[n!(n-1)!]$. Kubota

[10] proved the Springer's conjecture for $n = 3, 4$, and 5. Schober [14] obtained sharp bounds for B_{2n-1}, $1 \leq n \leq 7$. Recently, Kapoor and Mishra [9] found certain coefficient estimates for meromorphic starlike bi-univalent functions of order α. In the present paper we introduce some coefficient estimates for meromorphic strongly starlike and strongly Bazilevic bi-univalent functions.

2. Coefficient Estimates

To prove our theorems in this section we shall need the following two lemmas.

Lemma 2.1. If $p(z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$ is so that $Re(p(z)) > 0$ in Δ then $|c_n| \leq 2$.

The above lemma can be easily justified by using the Caratheodory Lemma (see Goodman [8] page 80) upon replacing $p(z)$ in Δ with $p(1/z)$ in $\mathbb{D}$.

Lemma 2.2. [7] If $g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \in \Sigma$, then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

A function $g \in \Sigma$ is bi-univalent strongly starlike of order α, $0 < \alpha \leq 1$, if

$$\left| \arg \left(\frac{zg'(z)}{g(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \Delta),$$

and

$$\left| \arg \left(\frac{wh'(w)}{h(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \Delta).$$

where the functions $h = g^{-1}$ is the inverse of g.

A function $g \in \Sigma$ is bi-univalent strongly Bazilevic of order $\alpha; 0 < \alpha \leq 1$ and type $\beta; 0 \leq \beta < 1$ if

$$\left| \arg \left(\left(\frac{z}{g(z)} \right)^{1-\beta} g'(z) \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \Delta)$$

and

$$\left| \arg \left(\left(\frac{w}{h(w)} \right)^{1-\beta} h'(w) \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \Delta)$$

where the function $h = g^{-1}$ is the inverse of g.

The class of Bazilevic functions was first defined and investigated by Bazilevic [2]. We note that bi-univalent strongly Bazilevic functions of order $\alpha; 0 < \alpha \leq 1$ and type zero are bi-univalent strongly starlike of order $\alpha; 0 < \alpha \leq 1$.

For $g \in \Sigma$ and $h = g^{-1}$Netanyahoo [12] proved that $|B_0| \leq 2$. We observe that

$$\left|\frac{B_0}{w} + \frac{B_1}{w^2} + \frac{B_2}{w^3} + \cdots\right| \leq \left|\frac{B_0}{w}\right| + \frac{\left|\sum_{n=1}^{\infty} \frac{B_n}{w^n}\right|}{|w|}.$$

Now the Cauchy- Schwarz inequality gives

$$\begin{aligned}
\left|\frac{B_0}{w} + \frac{B_1}{w^2} + \frac{B_2}{w^3} + \cdots\right| &\leq \left|\frac{B_0}{w}\right| + \frac{\left(\sum_{n=1}^{\infty} |B_n|^2 \sum_{n=1}^{\infty} \left|\frac{1}{w^n}\right|^2\right)^{\frac{1}{2}}}{|w|} \\
&\leq \left|\frac{B_0}{w}\right| + \frac{\left(\sum_{n=1}^{\infty} |B_n|^2 \sum_{n=1}^{\infty} \left|\frac{1}{2^n}\right|^2\right)^{\frac{1}{2}}}{|w|} \\
&\leq \left|\frac{B_0}{w}\right| + \frac{\left(\sum_{n=1}^{\infty} |B_n|^2\right)^{\frac{1}{2}}}{|w|} \\
&\leq \left|\frac{B_0}{w}\right| + \frac{\left(\sum_{n=1}^{\infty} n|B_n|^2\right)^{\frac{1}{2}}}{|w|}.
\end{aligned}$$

Applying Lemma 2.2 for $M = 3 < |w| < \infty$ we obtain

$$\left|\frac{B_0}{w} + \frac{B_1}{w^2} + \frac{B_2}{w^3} + \cdots\right| \leq \frac{2}{3} + \frac{1}{3} \leq 1.$$

We are now ready to state and prove our theorems.

Theorem 2.3. If $g \in \Sigma$ is bi-univalent strongly starlike of order $\alpha; 0 < \alpha \leq 1$, then

i) $|b_0| \leq \sqrt[2]{2\alpha(2-\alpha)}$,

ii) $|b_1| \leq \alpha$,

iii) $|b_2| \leq \frac{2}{9}\alpha\left[2\alpha^2 - 12\alpha + 13 + 3(2-\alpha)\sqrt[2]{2\alpha(2-\alpha)}\right]$.

Proof. If $g \in \Sigma$ is strongly starlike of order α, $0 < \alpha \leq 1$ then for some $p(z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$ where $Re\,p(z) > 0$ in Δ we can write

$$\frac{zg'(z)}{g(z)} = (p(z))^{\alpha}.$$

Comparing the corresponding coefficients of

$$\frac{zg'(z)}{g(z)} = 1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} - \frac{b_0^3 - 3b_0b_1 + 3b_2}{z^3} + \cdots$$

and

$$(p(z))^{\alpha} = 1 + \frac{\alpha c_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2}{z^2} + \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3 + \alpha(\alpha-1)c_1c_2 + \alpha c_3}{z^3} + \cdots$$

we obtain

$$\alpha c_1 = -b_0,$$

$$\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2 = b_0^2 - 2b_1,$$

and

$$\frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3 + \alpha(\alpha-1)c_1c_2 + \alpha c_3 = -\left(b_0^3 - 3b_0b_1 + 3b_2\right).$$

If moreover, $g \in \Sigma$ is bi-univalent strongly starlike of order α, $0 < \alpha \leq 1$, then for $h = g^{-1}$ there is exists a function $q(w) = 1 + \sum_{n=1}^{\infty} \frac{d_n}{w^n}$ with $Re q(w) > 0$ in Δ so that

$$\frac{wh'(w)}{h(w)} = (q(w))^{\alpha}.$$

Similarly, comparing the corresponding coefficients of $\frac{wh'(w)}{h(w)}$ and $(q(w))^{\alpha}$ we obtain

$$\alpha d_1 = -b_0,$$

$$\frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2 = b_0^2 + 2b_1,$$

and

$$\frac{1}{6}\alpha(\alpha-1)(\alpha-2)d_1^3 + \alpha(\alpha-1)d_1d_2 + \alpha d_3 = b_0^3 + 6b_0b_1 + 3b_2.$$

Now comparing the corresponding coefficient equations obtained above and using elementary algebraic manipulations, we observe that

$$c_1 = -d_1,$$

$$\begin{aligned} b_0^2 &= \frac{1}{2}\left(\frac{1}{2}\alpha(\alpha-1)(c_1^2+d_1^2)+\alpha(c_2+d_2)\right) \\ &= \frac{1}{2}\left(\alpha(\alpha-1)(c_1^2)+2\alpha(c_2)\right), \end{aligned}$$

$$4b_1 = \frac{1}{2}\alpha(\alpha-1)(d_1^2-c_1^2)+\alpha(d_2-c_2) = \alpha(d_2-c_2),$$

and

$$3b_0^3+9b_2 = \frac{1}{6}\alpha(\alpha-1)(\alpha-2)(d_1^3-2c_1^3)+\alpha(\alpha-1)(d_1d_2-2c_1c_2)+\alpha(d_3-2c_3).$$

Now, from Lemma 2.1, we notice that $|c_n| \leq 2$ and $|d_n| \leq 2$. Therefore,

$$\begin{aligned} |b_0| &\leq \sqrt[2]{\frac{1}{2}\left[\alpha(1-\alpha)|c_1|^2+2\alpha|c_2|\right]} \\ &\leq \sqrt[2]{\frac{1}{2}\left[\alpha(1-\alpha)(4)+\alpha(4)\right]} = \sqrt[2]{2\alpha(2-\alpha)}. \end{aligned}$$

Similarly,

$$|b_1| = \frac{1}{4}\alpha\,|d_2-c_2| \leq \frac{1}{4}\alpha\,(|d_2|+|c_2|) \leq \alpha.$$

Using the relations $d_1 = -c_1$ and $d_1^3 = -c_1^3$ for b_2 we obtain

$$\begin{aligned} 9b_2 &= \frac{1}{6}\alpha(\alpha-1)(\alpha-2)(d_1^3+2d_1^3)+\alpha(\alpha-1)(d_1d_2+2d_1c_2)+\alpha(d_3-2c_3)-3b_0^3 \\ &= \frac{1}{2}\alpha(\alpha-1)(\alpha-2)d_1^3+\alpha(\alpha-1)d_1(d_2+2c_2)+\alpha(d_3-2c_3)-3b_0^3. \end{aligned}$$

Using the estimate $|b_0| \leq \sqrt[2]{2\alpha(2-\alpha)}$ in conjunction with the facts that $|c_n| \leq 2$ and $|d_n| \leq 2$ we obtain

$$9|b_2| \leq 4\alpha(1-\alpha)(5-\alpha)+6\alpha+3\sqrt[2]{2\alpha(2-\alpha)}^{\,3}.$$

Or

$$|b_2| \leq \frac{2}{9}\alpha\left[2\alpha^2-12\alpha+13+3(2-\alpha)\sqrt[2]{2\alpha(2-\alpha)}\right].$$

■

Theorem 2.4. If $g \in \Sigma$ is bi-univalent strongly Bazilevic of order $\alpha; 0 < \alpha \leq 1$ and type $\beta; 0 \leq \beta < 1$ then

i) $|b_0| \le \sqrt{\dfrac{4\alpha(2-\alpha)}{(1-\beta)(2-\beta)}}$,

ii) $|b_1| \le \dfrac{2\alpha}{(2-\beta)}$,

iii) $|b_2| \le \dfrac{4\alpha(1-\alpha)(5-\alpha)+6\alpha}{3(3-\beta)} + \dfrac{4}{3}\alpha(2-\alpha)\sqrt{\dfrac{\alpha(2-\alpha)}{(2-\beta)(1-\beta)}}$.

Proof. If $g \in \Sigma$ is strongly Bazilevic of order $\alpha; 0 < \alpha \le 1$ and type $\beta; 0 \le \beta < 1$ then for some $p(z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$ where $Rep(z) > 0$ in Δ we can write

$$\left(\frac{z}{g(z)}\right)^{1-\beta} g'(z) = (p(z))^{\alpha}.$$

Comparing the corresponding coefficients of $(p(z))^{\alpha}$ and

$$\left(\frac{z}{g(z)}\right)^{1-\beta} g'(z) = 1 - \frac{(1-\beta)b_0}{z} + \frac{(2-\beta)\left((1-\beta)b_0^2 - 2b_1\right)}{2z^2} - \frac{(3-\beta)\left((1-\beta)(2-\beta)b_0^3 + 6(1-\beta)\,b_1b_0 + 6b_2\right)}{6z^3} + \cdots$$

we obtain

$$-(1-\beta)b_0 = \alpha c_1,$$

$$\frac{1}{2}(2-\beta)\left((1-\beta)b_0^2 - 2b_1\right) = \frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2,$$

and

$$-\frac{1}{6}(3-\beta)\left((1-\beta)(2-\beta)b_0^3 - 6(1-\beta)\,b_1b_0 + 6b_2\right) =$$

$$\frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3 + \alpha(\alpha-1)c_1c_2 + \alpha c_3.$$

If moreover, $g \in \Sigma$ is bi-univalent strongly Bazilevic of order $\alpha; 0 < \alpha \le 1$ and type $\beta; 0 \le \beta < 1$ then for $h = g^{-1}$ there exists a function $q(w) = 1 + \sum_{n=1}^{\infty} \frac{d_n}{w^n}$ with $Req(w) > 0$ in Δ so that

$$\left(\frac{w}{h(w)}\right)^{1-\beta} h'(w) = (q(w))^{\alpha}.$$

Similarly, comparing the corresponding coefficients of $(q(w))^{\alpha}$ and

$$\left(\frac{w}{h(w)}\right)^{1-\beta} h'(w) = 1+\frac{(1-\beta)b_0}{w}+\frac{(2-\beta)((1-\beta)b_0^2+2b_1)}{2w^2} +\frac{(3-\beta)((1-\beta)(2-\beta)b_0^3+6(1-\beta)b_0b_1+6b_2)}{6w^3}+\cdots$$

we obtain

$$(1-\beta)b_0=\alpha d_1,$$

$$\frac{1}{2}(2-\beta)\left((1-\beta)b_0^2+2b_1\right)=\frac{1}{2}\alpha(\alpha-1)d_1^2+\alpha d_2,$$

and

$$\frac{1}{6}(3-\beta)\left((1-\beta)(2-\beta)b_0^3+6(1-\beta)b_0b_1+6b_2\right)=$$

$$\frac{1}{6}\alpha(\alpha-1)(\alpha-2)d_1^3+\alpha(\alpha-1)d_1d_2+\alpha d_3.$$

Once again, comparing the corresponding coefficient equations obtained above and using elementary algebraic manipulations, we observe that

$$c_1=-d_1,$$

$$\begin{aligned}(2-\beta)(1-\beta)b_0^2 &= \frac{1}{2}\alpha(\alpha-1)(c_1^2+d_1^2)+\alpha(c_2+d_2)\\ &= \alpha(\alpha-1)c_1^2+\alpha(c_2+d_2),\end{aligned}$$

$$2(2-\beta)b_1=\frac{1}{2}\alpha(\alpha-1)(d_1^2-c_1^2)+\alpha(d_2-c_2)=\alpha(d_2-c_2),$$

and

$$\frac{1}{6}(3-\beta)\left[2(1-\beta)(2-\beta)b_0^3+12b_2\right]=$$

$$\frac{1}{6}\alpha(\alpha-1)(\alpha-2)(d_1^3-c_1^3)+\alpha(\alpha-1)(d_1d_2-c_1c_2)+\alpha(d_3-c_3)=$$

$$\frac{1}{3}\alpha(\alpha-1)(\alpha-2)d_1^3+\alpha(\alpha-1)d_1(d_2+c_2)+\alpha(d_3-c_3).$$

Therefore, for b_0 we obtain

$$\begin{aligned}(1-\beta)(2-\beta)|b_0|^2 &= \left[\alpha(1-\alpha)|c_1|^2+\alpha(|c_2|+|d_2|)\right] \\ &\leq 4\alpha(1-\alpha)+4\alpha = 4\alpha(2-\alpha)\end{aligned}$$

or

$$|b_0| \leq \sqrt{\frac{4\alpha(2-\alpha)}{(2-\beta)(1-\beta)}}.$$

In regards to b_1 we obtain

$$2(2-\beta)|b_1| = \alpha|d_2-c_2| \leq \alpha(|d_2+|c_2|) \leq 4\alpha$$

or

$$|b_1| \leq \frac{2\alpha}{(2-\beta)}.$$

For the coefficient b_2 we have

$$b_2 = \frac{\alpha(\alpha-1)(\alpha-2)d_1^3+3\alpha(\alpha-1)d_1(d_2+c_2)+3\alpha(d_3-c_3)-(1-\beta)(2-\beta)(3-\beta)b_0^3}{6(3-\beta)}.$$

Upon using the bounds for $|b_0|$, $|c_n|$ and $|d_n|$ we obtain

$$\begin{aligned}|b_2| &\leq \frac{8\alpha(1-\alpha)(2-\alpha)+24\alpha(1-\alpha)+12\alpha+(1-\beta)(2-\beta)(3-\beta)|b_0|^3}{6(3-\beta)} \\ &\leq \frac{4\alpha(1-\alpha)(5-\alpha)+6\alpha}{3(3-\beta)}+\frac{4}{3}\alpha(2-\alpha)\sqrt{\frac{\alpha(2-\alpha)}{(2-\beta)(1-\beta)}}.\end{aligned}$$

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