International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 4 (2021), pp. 323-332

The problem of tangency to three non-homothetic conics

H. Isawi* [a], S. Dahabreh [a], D. Altarawneh [a], H. Alsawalqa [a]

[a] *Department of Architecture – School of Engineering, The University of Jordan*

Abstract

The aim of this research is to formulate a procedure for determining the four copies of conics tangents to three established conics, both homothetic and generic, and to illustrate and justify in space the elements that originated the geometric constructions carried out in the plan. In the process of modeling a tangential connection between two (or three) conic surfaces, we often feel disoriented when we face the problem of tangency between non-homothetic conics. To solve the problem, topics such as involution between coincident conics were treated; the inversion of a quadric cone; the pole line and the corresponding polar plane; the cyclic transformations defined by three coplanar conics; and the polar cone as the locus of the pole lines of three conjugated cones.

Keywords: Tangency, Non-homothetic conics, involution, polarity, Cyclical transformation, Pole line, Polar cone, Inverse cone, Envelope curves, Cyclide.

I. Introduction

The described geometric procedure obtained as a result of this research allows not only to determine the possible conics tangent to three assigned homothetic conics but also to three established conics different from each other, provided that they are coplanar and corresponding to each other two by two.

The research problem can be considered as an extension of the so-called "Apollonius problem" [1], particularly concerned with the determination of all possible tangent circles to three given circles, including the point and the line as particular cases of circumference [2]. Euclid solved this particular problem in which these circles degenerate into points or lines. As for the more general case, of the three circles, the problem was solved by the Belgian mathematician Adrian van Roomen [3] through the determination of the points common to the geometric loci of the centers of the circumferences tangent to those assigned. The Roomen procedure can be adopted, much more easily today, not only in cases of tangency between circumferences but also in all those cases in which three homothetic conics are given.

For example, three homothetic conics **Δ, θ, and Φ** are assigned (fig. 1), belonging to a horizontal plane $\boldsymbol{\pi_1}$, and we want to determine the eight conics tangent to them. **Δ, θ, and Φ** are considered the bases of three cones **u, v, and w**, conjugated to each other, two by two [4]. Between each copy of these bases, there is a homothetic correspondence, in which the corresponding lines are parallel to each other and pass through the already known corresponding points that are the centers of the same bases.

The geometric locus of the centers of the conics are tangents of two of the three given conics, for example, **Δ** and **θ** is determined, in this case (fig. 1), as the horizontal orthographic projection (top view) of the intersection between the cones **v** and **w**, which have **Δ** and **θ** as bases.

Similarly, once the other geometric loci of the other two pairs **Δ, Φ** and **θ, Φ** are determined, the centers of the eight conics are easily found as common points to the same loci (Fig. 1). Regarding this particular case, it is important to check the more general procedure formulated in the last pages of this research (fig. 13).

It should be noted that this research concentrates on the study of the general problem of tangency between non-homothetic conics. For a more detailed discussion of listing and classifying the various cases of tangency between homothetic conics, Riccardo Migliari's article [5] covered adequately those particular cases.

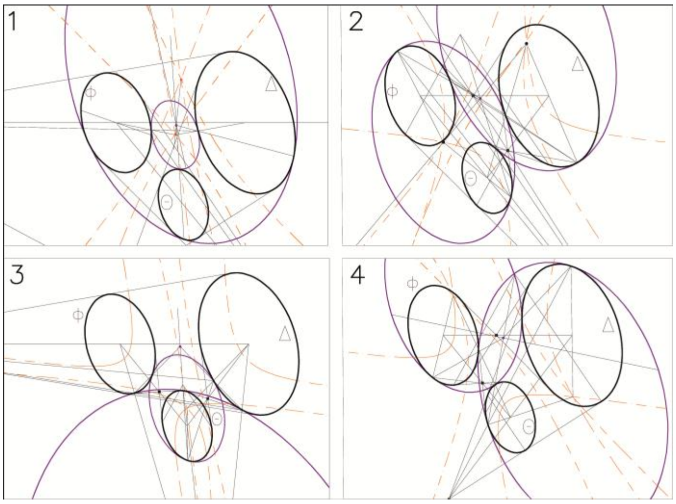


Figure 1*: The determination of the centers of the eight tangent conics to three assigned homothetic conics.*

II. Methodology

Faced with the immediacy of the operations designed to determine the elements of correspondence between two assigned homothetic conics, due to the fact that the centers are already corresponding points, it is impossible to find these elements when the assigned conics are generic. To overcome this non-existence and to be able to analyze the possible differences between the two cases, we have established that the three conics $\boldsymbol{\Delta_U}$, $\boldsymbol{\Delta_V}$, and $\boldsymbol{\Delta_W}$ are the bases of three cones **u, v,** and **w**, which, in addition to being conjugated to each other [6] two by two, also have in common the same conic **Δ**, belonging to the generic plane α (Fig.2, 3). This way, by determining the intersection of the three cones **u, v,** and **w**, two by two, we will have, in addition to **Δ** three other conics, **Φ, X,** and **Ψ**. For example, **Φ** is involutory of **Δ** with respect to the vertices of the two cones **v** and **w**, which produced this involution.

In order to verify this involution, just imagine projecting, respectively, one of the two conics, for example, **Δ** on the plane of **Φ**, from the vertices **v** and **w**, to notice that the projection of **Δ** coincides with **Φ** and vice versa. Therefore, the corresponding points of **Δ** and **Φ** correspond in a double way. The center of this involution is the point of intersection of the line passing through the vertices of **v** and **w**, with the plane of **Φ**.

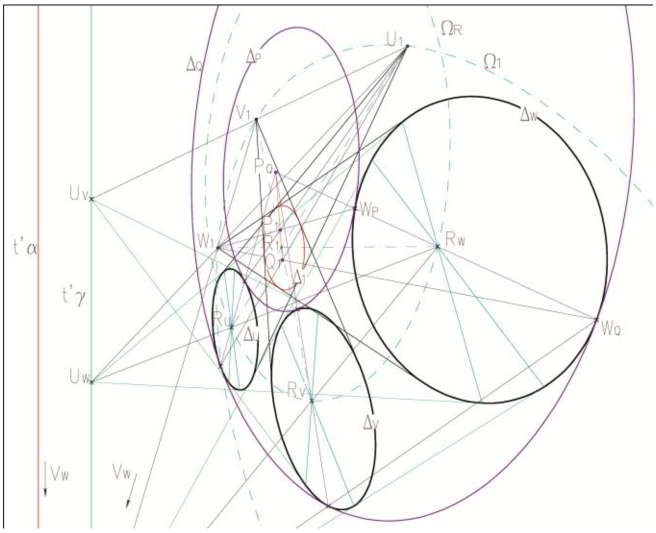


Figure 2*: the determination of the first copy of conics tangent to the established three generic conics*

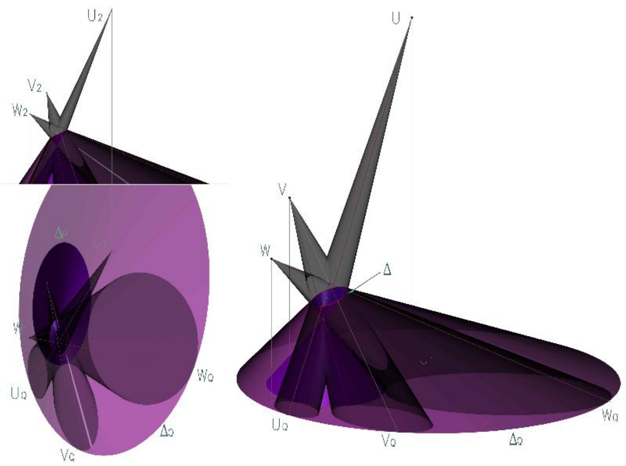


Figure 3*: 3D modeling of the situation present in figure 2*

I.III Intersections between cones having a conic in common

The intersection curve between two quadric cones [7] is a skew curve [8] that, in most cases, can be formed by a single branch (monogrammic) or by two branches (diagrammatic) [9]. This is determined through the use of a sheaf of auxiliary planes that have in common the line joining the vertices of the same cones. Each of these planes can dissect these cones according to two generatrices. Any points common to the generatrices of each section plane are the points of the intersection curve.

In the case discussed in this research, the three established cones already have the same conic in common, so the second branch of the intersection of each copy of these cones is also a conic. In this regard, it should be noted that between the infinite auxiliary planes of the mentioned sheaf, there are two planes that are tangents to both cones considered. It follows that the straight line passing through the two points common to these generators, of each tangency plane, identifies the axis of the mentioned involution between the coincident projections of two conics considered [10, 11]

The axis of the homological correspondence that exists, two by two, between the bases of the three established cones, for example between **v** and **w** (Fig. 2), is the horizontal trace of the plane **α** where the objective conic **Δ** lies. The center of this homology is the horizontal trace of the line joining the vertices of the two cones considered (**v** and **w)**.

I.IV The condition of tangency between the bases of five cones with a conic in common

Once we have determined the three conics; **Φ, X,** and **Ψ**, common, two by two, to the three established cones; **u, v,** and **w**, we proceed to identify their two common points **P** and **Q**. As these two points are the vertices of the two other cones; **p** and **q,** which have, in turn, the conic **Δ** in common, we can conclude that their bases $\mathbf{\Delta_P}$ and $\mathbf{\Delta_Q}$ are tangent to the bases $\mathbf{\Delta_U}$, $\mathbf{\Delta_V}$, and $\mathbf{\Delta_W}$ of the cones **u, v,** and **w**.

The justification of this tangency is due to the fact that the vertices **P** and **Q** of the cones **p** and **q** have been determined, respectively, as points common to three generatrices of the established cones; **u, v,** and **w**. In addition, by sectioning all five cones: **u, v, w, p,** and **q** with any plane, we find that the sections of **p** and **q** are tangents to those of the three cones **u, v,** and **w**. By assuming that the section plane coincides with **γ** - identified by the vertices of **u, v,** and **w**- we have, in addition to the three degenerate conics, which are these vertices, also a conic **Ω**, which represents the two coincident sections of the two cones **p** and **q**. In other words, **Ω** is obtained as coincident projections of **Δ** from the vertices of **p** and **q** on the **γ** plane.

The points of tangency, such as $\mathbf{W_P}$ and $\mathbf{W_Q}$, in the intercurrent homology, two by two, between the bases of **p** and **q** and those of **u, v,** and **w**, are aligned with the same center $\mathbf{P_Q}$, which is determined, in this case, as the horizontal trace of the line **r** (fig. 2), which is common to the Sheaf of the three planes where the aforementioned three conics **Φ, X,** and **Ψ** lie respectively. Instead, the axis, as already stated, is the horizontal trace of the plane **α** where the established conic **Δ** lies. Furthermore, among the planes of that Sheaf, there are also those that have as their first traces the straight lines passing through the sought points of tangency, between the bases of **p** and **q** and those of **u, v,** and **w**.

In this way, we have partially described and justified a spatial situation (fig.3) that produced one of the four copies of the researched conics. These act as bases of the two cones **p** and **q**, which in addition to having the same conic **Δ** in common; also have the vertices as common points, respectively, to the three established cones **u, v,** and **w**. Furthermore; their bases, **Δ_P** and **Δ_Q**, are tangents to those of **u, v,** and **w**.

In order to facilitate the following reasons, we have used, respectively, the terms: main cone and branched cone, in order to be able to quickly distinguish between the three established cones, such as **u, v,** and **w**, and those determined subsequently, such as **p** and **q** (which have, respectively, a common generator with such established cones). We have also used the terms: direct cone and inverse cone, in order to distinguish between the cone that has by sections two conics that corresponded in a direct way and the one that corresponded inversely.

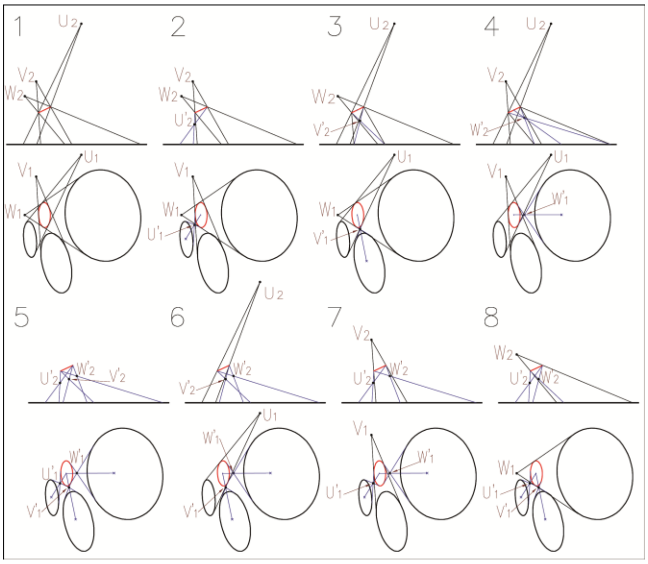


Figure 4: *the schemes of the eight combinations of the vertices of three established cones, u, v, w with respect to their common conic Δ.*

I.V The direct and inverse correspondences between the sections of a cone

Considering that the vertex of each of the three main cones can be combined between two positions: direct or inverse, with respect to two distinct sections of the same cone, the total number of combinations of such cones is eight, which have been divided into two groups (Fig. 4). The reason for this division arises from the fact that by adopting one of the four provisions present in the first row or its relative inverse, in the second row, the same identical results are obtained in both cases. For example, by adopting both the combination number 2 and number 6, we have two copies of coincident and tangent conics, respectively to the three established conics.

The basic concept to which the various procedures refer, depends in particular on the fact that by sectioning, for example, two cones **v** and **p** (Fig. 5-1), having in common a section **Δ**, and also a generatrix **g**, with the same plane, π_1, we will have as section, two conics **Δ_V**, and **Δ_P** that are tangents to each other at the point of intersection **V_P** of **g** with π_1. Following, the homology between **Δ_V Δ_P** has the point of tangency **V_P** as its center, between **Δ_V** and **Δ_P**, and has its axis as the intersection line of α, where **Δ** lies with π_1.

The justification for this coincidence arises from the fact that the five cones obtained both from a given combination and from its relative inverse, all have the same conic section in common. For example (fig. 5-2), the inverse v ' and p' of the two cones **v** and **p**, which have in common both the conical section **Δ** and the generatrix **g**, are two cones which have, in turn, in common over **Δ**, also a generatrix **g'**, which is the inverse of **g**. By dissecting both the cones **v** and **p**, and the relative inverses **v'** and **p'**, we conclude that the sections of **v** and **v'** in addition to being coincident with each other, are also tangents to the coincident sections of **p** and **p'**.

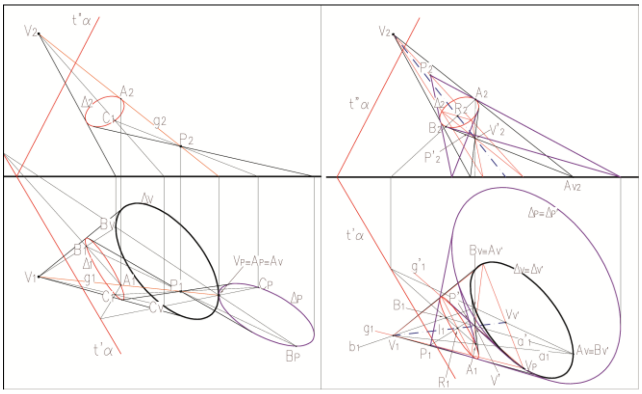


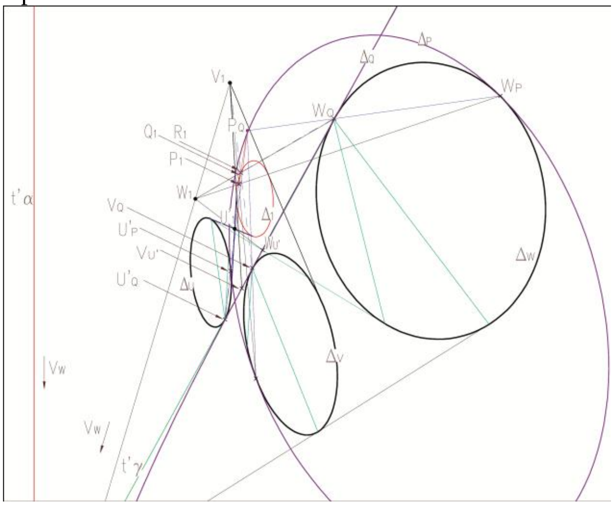
Figure 5*: The drawing on the left (1) there are the projections of two cones v and p, in which the vertex of one of which belongs to the generatrix of the other. The drawing on the right (2) illustrates the determination of both the inverse v' of the main cone v and the inverse p' of the branched p of v.*

There are various homologies that exist, two by two, between the same bases of the three main cones and between the same bases of those of the two branched cones (revise fig. 2). Since all these five cones have in common the same conic **Δ**, the axis of the various homologies is always the horizontal trace tα of the plane **α** where **Δ** lies. Instead, the center varies according to the mutual positions of the vertices of these cones with respect to the plane of homology π_1, it is determined as the horizontal trace of the line joining the vertices of the two cones considered. For example (fig. 5-1), the homology between the bases of the cones **v** and **p**, has its axis as **t'α** and its center as the point of tangency **V_P** between the bases **Δ_V** and **Δ_P** of **v** and **p**.

I.VI The inversion of a quadric cone

The operations designed to determine both the inverse **v'** of the main cone **v** and the inverse **p'** of the branched **p** of **v** (fig. 5-2) are the following:

- The vertex **V'** of **v'** is determined as the intersection point of the two generatrices **b'** and **c'**, which are obtained by inverting the correspondence between the projections of the two sections **Δ_V** and **Δ** of **v**. To this end, the pole **$V_{V'}$** of **t'α** with respect to **Δ_V** is determined; and is joined with **V**, thus, obtaining the line **l**, for which plane **β** is passes, which section **v** according to the two generatrices **b c**. The intersection which with **Δ_V** and **Δ** identifies a copy of corresponding points **A**, **A_V** and **B**, **B_V**. The center and axis of this correspondence are respectively **V** and **tα**. Since the center **V** is on the same side with respect to the corresponding points, the correspondence is said to be direct, otherwise, it is inverse when **V** is in an intermediate position.



- **Figure 6**: the determination of the second copy of the conics tangent to the established three generic conics Δu, Δv, and Δw

- The sought vertex **V'** of the cone **v'** is determined as the intersection point of the lines **b' c'**, which are the inverse generatrices of **b c**. The determination of these generatrices **b' c'** occurs by reversing the alignment between the copies **A_AV** and **B_Bv** of corresponding points. That is one of the corresponding points, for example, $\mathbf{A_V}$ is joined with the corresponding **B** of the other point $\mathbf{B_V}$ and vice versa.
- Finally, the vertex **P'** of the inverse **p'** of the cone **p** is determined as the intersection point of the inverse generatrix **g'** of g with the line joining the vertex **P** of the branched cone **p** with the center of the inversion **R** of the two cones **v** and **v'**. This center is determined as the intersection point of **l** with **α**.

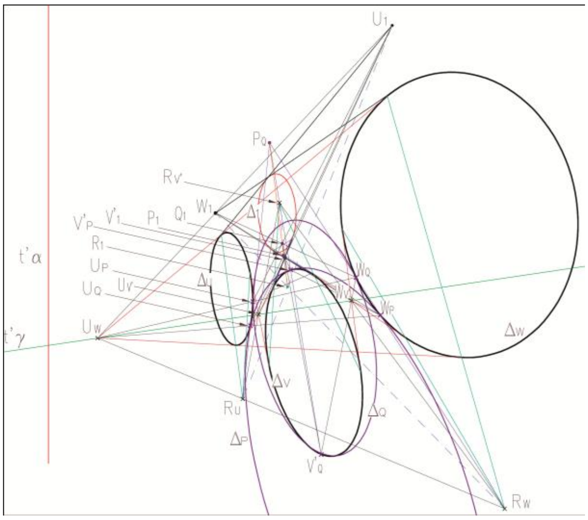


Figure 7*: the determination of the third copy of the conics tangent to the three established generic conics Δu, Δv, and Δw*

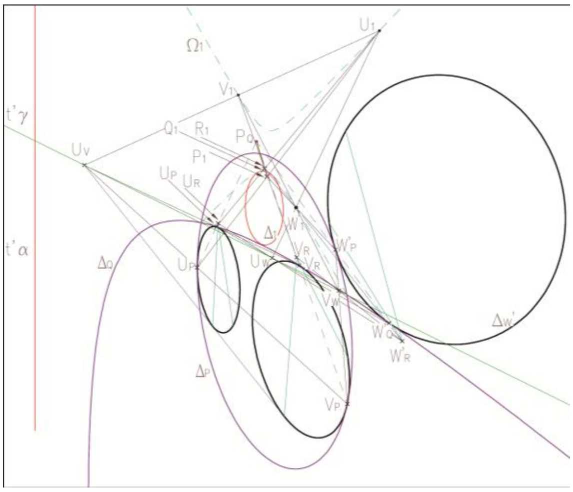


Figure 8*: the determination of the fourth copy of the conics tangent to the three established generic conics Δu, Δv,and Δw*

I.V1I The difference between direct and inverse tangency

It should be noted that in the various solutions (Fig.2, 6, 7, 8) the bases of the branched cones include one or more bases of the three main cones, or exclude them. This fact depends on the reciprocal position of three elements which are the vertex of a branched cone and that of the main cone and their common section **Δ**. According to the type of perspectivity, direct or inverse, which occurs, respectively, between **Δ** and the bases of these cones, two cases can generally arise: when the vertices of these cones act as centers of two direct perspectives, or inverse, the base of the branched cone includes that of the main cone; otherwise it excludes it when one of the two perspectives is direct and the other is inverse. For example (fig. 11), the basis $\mathbf{\Delta_Q}$ of **q** includes $\mathbf{\Delta_W}$ of **w** because they are obtained respectively from two direct perspectives. Instead, the two bases $\mathbf{\Delta_W}$ and $\mathbf{\Delta_P}$ are mutually exclusive because they have been obtained from two different perspectives.

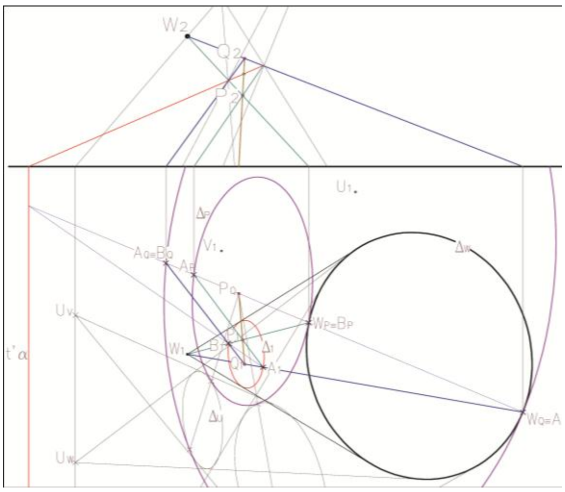


Figure 9*: According to the type of perspectives that produced two projections of the same conic, it can happen that when both perspectives are direct (or inverse), one projection includes the other; otherwise, they are mutually exclusive when one perspective is direct and the other is inverse.*

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 4 (2021), pp. 323-332

In each of the cases dealt with seen in fig. (11,12), it can be noted that the poles $\mathbf{R_U}$, $\mathbf{R_V}$, and $\mathbf{R_W}$ of **t'α**, with respect to the three established conics; Δ_U, Δ_V, and Δ_W, are projections of the same point **R** from vertices of the main cones; **u, v,** and **w** on π_1. Hence, **R** acts as the objective pole of these projections. Furthermore, point **R** is the pole of the polar obtained as a common line to the **α** and **γ** planes with respect to the conic **Δ**. Where α is the plane where **Δ** lies and **γ** is the one identified by the vertices of **u, v,** and **w**. For example, by joining **R** with the pole $\mathbf{V_P}$ of the base Δ_V of **v** we have d as the pole line [12] of the cone v with respect to the horizontal trace (or the first trace) of **γ**. The pole line **d** has the property of being the locus of the poles [13] of the polar obtained, respectively, as lines of intersection of the planes sectioning the cone v, with respect to the conic sections of these planes. In other words, by sectioning both; **γ** according to line **r**, and the cone **v** according to conic **Δ**, with any plane, **α**, we have in any case that the pole **R** of **r** with respect to **Δ** belongs to the pole line **d**. In this regard, the coincident projections in $\mathbf{P_Q}$ of the objective pole **R** from the vertices of the two branched cones **p q** on π_1 is the center of the cyclic transformation that not only envelops the bases of **u, v,** and **w**, but also includes all the bases of those cones which have, respectively, as vertices, the points belonging to **γ**, and as a common section the same conic **Δ**.

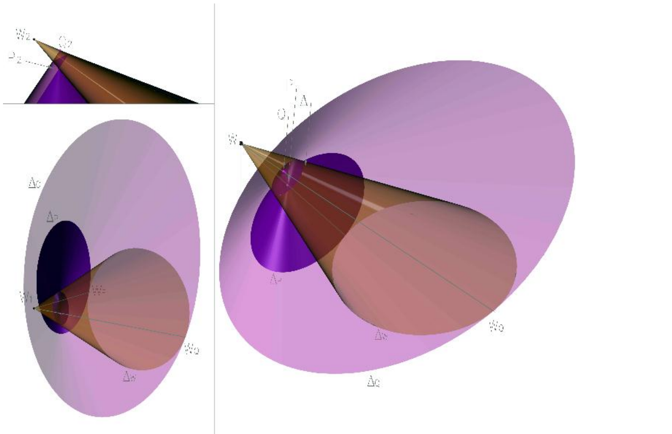


***Figure 10**: 3D modeling of the situation present in figure 9*

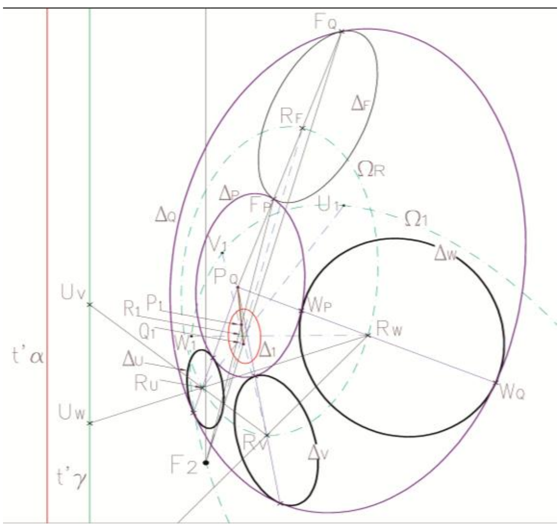


***Figure 11**: The cyclic transformation of the three established conics; ΔU, ΔV, and ΔW, has the point PQ as its center, the conic ΩP as its director axis, and ΔP, ΔQ as the envelope lines.*

In this case, specifically, the bases are horizontal cross-sections of three cones (fig. 13), the pole lines of these cones are parallel to each other, and therefore, the place of these pole lines is a cylinder and not a cone, unlike the previous cases.

In conclusion, it can be stated that the invariant in each of the four cyclic transformations is not the locus of the centers of the enveloped conics; Δ_U, Δ_V, and Δ_W and their transformed conics as it occurs in the case of homothetic ones; but it is that of the poles of **t'γ** (the first trace of the plane **γ** identified by the vertices of the three main cones; **u, v,** and **w**) with respect to the mentioned conics.

I.VIII Results

In all four transformations, there are 24 points of tangency between the assigned conics Δ_U, Δ_V, and Δ_W and the four copies of the determined ones (fig. 14). These points of tangency are aligned, two by two, with the same point $\mathbf{P_Q}$, which acts as the center of a star of four sheaves of planes, which act respectively as support of the lines passing through the vertices of four copies of the branched cones.

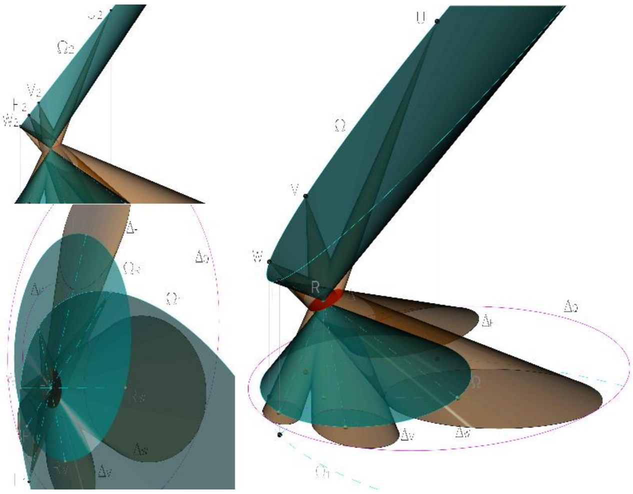


***Figure 12**: 3D modeling of the situation in figure 11*

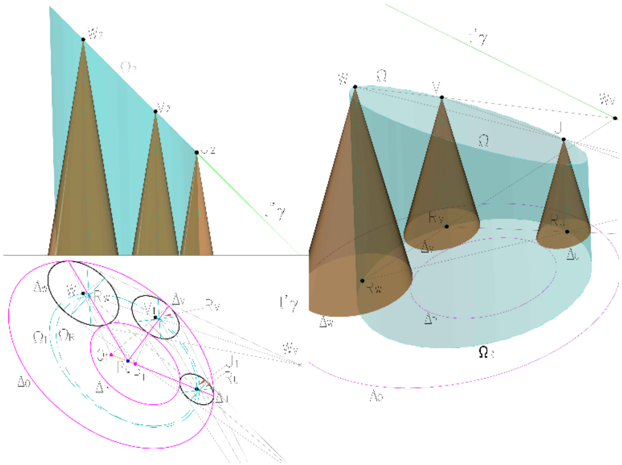


***Figure 13**: the polar cylinder of the three cones u, v, and w, has as generatrix the pole lines of these cones with respect to the corresponding plane γ identified by the vertices U, V, and W of u, v, and w. So the section of this cylinder with $\pi 1$ is a conic that acts as a locus of the poles of the first trace of γ with respect to the bases Δ_U, Δ_V, and Δ_W of u, v, and w.*

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 4 (2021), pp. 323-332

From the section of such a star with $\pi 1$, a sheaf of lines is obtained, which has the point P_Q as its center. 12 straight lines of this sheaf are respectively the traces of the planes that pass through the vertices of 20 cones which include 12 main cones (9 direct and 3 inverse) in the four combinations and 4 copies of the branched cones.

The other lines of the same sheaf, with reference to the said transformations, are respectively the traces of the planes that pass through the vertices of both a copy of branched cones and of other cones conjugated to the main ones.

The other straight lines of the sheaf are the traces of the planes that pass, respectively, through the vertices of the branched cones and through those of the conjugated cones to the main ones. All these conjugated cones have the vertices belonging to the same **γ** plane (identified by the vertices of the three main cones), and also have the same conic in common. Therefore, their bases are enveloped, respectively, by those of the four copies of branched cones.

In consideration of the fact that the four transformations are defined, respectively, by the three established conics, it is possible to pass from one transformation to another, without interruption, through one of these conics.

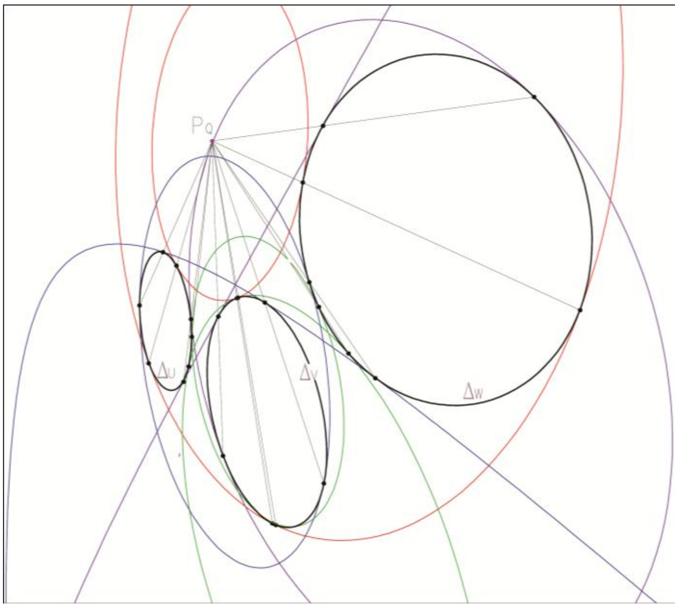


Figure 14*: the grouping of the four determined copies of conics tangent to the three established conics ΔU, ΔV, ΔW. In which we see that the 24 points of tangency are aligned, two by two, with the same point PQ*

III. The classification of cyclic transformations

In order to classify the cyclic transformation **G** (fig. 15) between coplanar conics, the following considerations should be kept in mind:

- **A** cyclic transformation **G** is identified whether three enveloped conics; $\mathbf{\Delta_U}$ $\mathbf{\Delta_V}$ and $\mathbf{\Delta_W}$, or two enveloping conics; Δ_P and Δ_Q, are known
- These five conics are considered as bases of as many cones, that have the same conic **Δ** in common, as possible, In which $\mathbf{\Delta_U}$, $\mathbf{\Delta_V}$, and $\mathbf{\Delta_W}$ indicate the bases of the three enveloped cones v, u, and w and $\mathbf{\Delta_P}$, $\mathbf{\Delta_Q}$ of the two enveloping cones **p** and **q**. Hence, $\mathbf{\Delta_P}$ and $\mathbf{\Delta_Q}$ are tangents to $\mathbf{\Delta_U}$, $\mathbf{\Delta_V}$, and $\mathbf{\Delta_W}$.
- The vertices **P Q** of the enveloping cones **p** and **q** are determined as common points of the enveloped cones **v, u,** and **w**.
- The two cones **p** and **q** which have in common a conic **Δ** intersect according to another conic **Ω**, which represents the locus of the vertices of the enveloped cones and is determined using a sheaf with an axis **d** (that passes through the vertices **P, Q** of **p, q**). Furthermore, it should be noted that **Ω** is the locus of the poles of degenerate conics, which are the vertices of the enveloped cones.
- Between the two conics **Δ** and **Ω**, common to two envelope cones **P** and **Q**, there is an involutive correspondence. Its axis is the intersection line of the two planes **α** and **γ**, where **Δ** and **Ω** lie, respectively. Instead, the center of this involution is the point of intersection **R** of the plane of **Δ** with the pole line **d**.
- The director axis $\mathbf{\Omega_R}$ of **G** is the geometric locus of the poles; $\mathbf{U_{R1}}$, $\mathbf{V_{R1}}$, and $\mathbf{W_{R1}}$ of the polar line (**t'γ**) with respect to the enveloped conics $\mathbf{\Delta_U}$, $\mathbf{\Delta_V}$, and $\mathbf{\Delta_W}$. These poles are determined as projections, on $\mathbf{\pi_1}$, of the vertices **U, V, and W**, of the enveloped cones from the vertex **R** of the polar cone **ρ**.
- The vertex R of the polar cone **ρ** is determined as the intersection of the line d with the plane α, where the objective conic **Δ** lies.
- The axis Ω_R is determined as a section of **ρ** with the π_1 plane, where the enveloped conics lie.

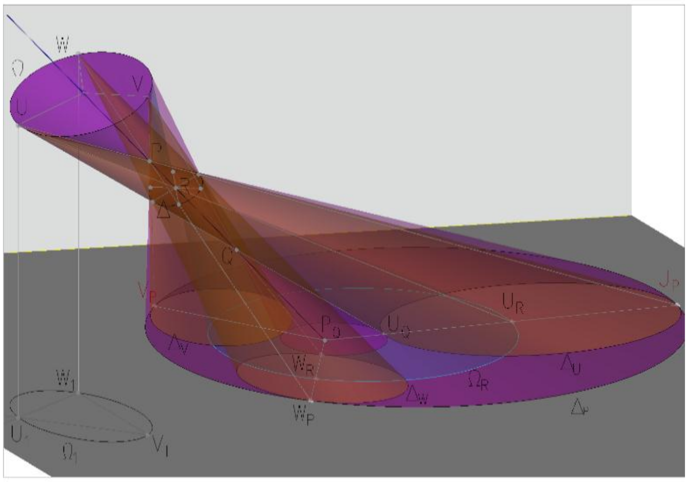


Figure 15*: an elliptic cyclic transformations*

Therefore, according to the position of the frontal plane $\boldsymbol{\pi_1^0}$ passing through **R**, with respect to the conic **Ω**, a cyclic transformation **G** can be classified as follows:

- Elliptic, when the plane $\boldsymbol{\pi_1^0}$, is external with respect to **Ω**. In this case, the base $\mathbf{\Omega_R}$ of the polar cone **ρ** is formed by all proper points
- Parabolic, when $\boldsymbol{\pi_1^0}$ is tangent Ω. In this case Ω_R has an improper point (a single point at infinity).
- Hyperbolic, when $\boldsymbol{\pi_1^0}$ is secant Ω. For which Ω_R has two improper points (two points at infinity).

The previous classification can also be applied to cyclids that envelop conjugated surfaces.

IV. A construction's modality of a cyclide with conjugated enveloped surfaces

A cyclic transformation in given (fig. 16), identified by three conics, $\boldsymbol{\Delta_U}$, $\boldsymbol{\Delta_V}$, and $\boldsymbol{\Delta_W}$, coplanar and non-homothetic, and by a copy of conics; $\boldsymbol{\Delta_P}$ and $\boldsymbol{\Delta_Q}$, tangent to the first ones. We want to model a cyclide **K** in which the given conics act, respectively, as bases of enveloped and enveloping surfaces.

It is established that the pole lines of all these surfaces are parallel to each other. It is decided that the corresponding plane γ of these pole lines passes through the line **tγ** and the point **U**, which can be decided on a pole line of the enveloped surfaces. For example, on **b** of **u.** The first trace **tγ** of the γ plane represents the polar of the poles with respect to the enveloped conics; $\boldsymbol{\Delta_U}$, $\boldsymbol{\Delta_V}$, and $\boldsymbol{\Delta_W}$.

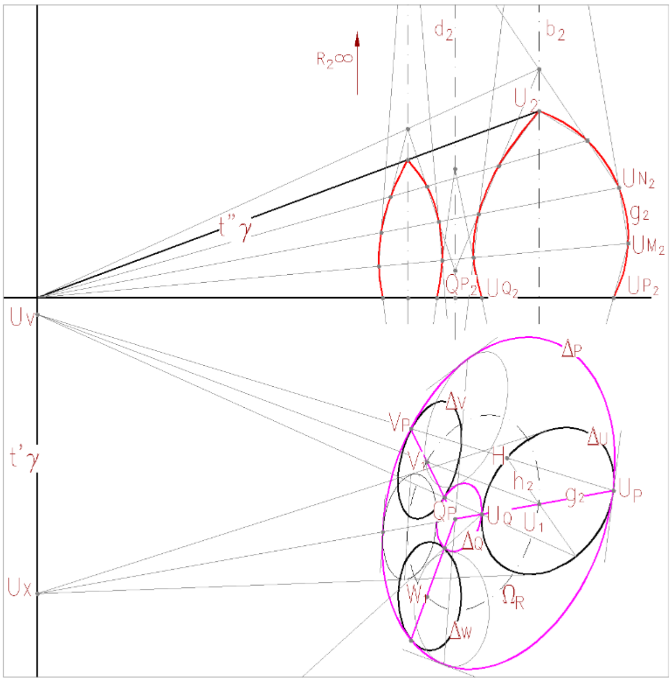


Figure 16*: the orthogonal projections of significant elements of a Cyclide enveloped conjugated surfaces*

Since the bases of the enveloped surfaces are non-homothetic conic, therefore, also the meridian generatrices of such surfaces are not homothetic. For example, if we decide that the generatrix **g** of **u** is an arc of the circumference, the other generatrices of **u** are not arcs of circles, but arcs of ellipses, which are determined, respectively, as, projections of g from centers belonging to **tγ**, on a sheaf of planes passing through b. For example, to determine the $\boldsymbol{U_V}$ center of the correspondence between two generatrices, **g h**, of **u**, we join two corresponding points, $\boldsymbol{U_P}$ **H**, of $\boldsymbol{\Delta_U}$, with a line that meets **t'γ** in the sought $\boldsymbol{U_V}$ center. Similarly, the meridian generatrices of the other enveloped surfaces are determined. For example, using the $\boldsymbol{U_V}$ center that we have just determined, we can project g onto the meridian plane of **v**, which passes through $\boldsymbol{U_P}$.

Instead, the directrices of the enveloped surfaces are determined as common sections of the consecutive series of auxiliary cones. These directrices are obtained as sections, of the auxiliary cones, with a sheaf of planes passing through **t'γ**.

To determine, for example, the directrices of **u**, proceed as follows: Divide the arc g into equal parts, and the division segments are prolonged until meeting the straight pole **b**. Thus, they identify the vertices of as many cones. The first cone m has the conic $\boldsymbol{\Delta_U}$ as its base and the point M as its vertex, which is determined as the intersection point of b with the extension of the $\boldsymbol{U_P}$**-**$\boldsymbol{U_M}$ segment, where $\boldsymbol{U_P}$ is the point that belongs to $\boldsymbol{\Delta_U}$, and the other point $\boldsymbol{U_M}$ belongs to the sought second directrix of **u**. The determination of which is obtained by sectioning the first cone m with the plane passing through the point $\boldsymbol{U_M}$ and the line **t'γ**. The third directrix of u is obtained by sectioning the third cone n with the plane passing through $\boldsymbol{U_M}$ and **t'γ**. The vertex **N** of n is the intersection point of b with the extension of the segment $\boldsymbol{U_M}$**-**$\boldsymbol{U_N}$, where point $\boldsymbol{U_N}$ refers to the mentioned division of **g**. Similarly to the previous steps, the other directrices of **u** are determined.

Keep in mind that the sheaf of planes that pass through **tγ** sections the surfaces, both enveloping and enveloped, according to a series of cyclical transformations. The centers of which are determined, respectively, as the intersection points of the pole line **d** with the planes of the same sheaf. Furthermore, the straight pole d of the envelope surface, which in addition to representing the locus of the poles of **tγ** with respect to the various directrices of the cyclide **K**, is also the locus of the vertices of the auxiliary cones that approximate the surface of **k**.

The line **d**, in this case, represents the axis of a rotation, but with reference to all cases of surface of revolution, whether circular, elliptical, parabolic, or hyperbolic; the meridian generatrices of the enveloped surfaces coincide, respectively, with those of the enveloping ones.

For the construction of a cyclide, in general, it should be noted that there are two types of sheaves of planes on which lie the main generatrices and directrices of both the enveloping and enveloped surfaces. The first type is represented by a set of sheaves of meridian planes that pass respectively through the pole lines of these surfaces; the second type passes through the polar line of the poles with respect to the directrices of the same surfaces.

V. Results and Discussions

At this point, and after having solved the problem proposed by this research: the determination of all the possible conics tangent to three established coplanar and non-homothetic conics, it would be important to mention the possible applications of this solution in the field of architecture. Therefore, the descriptive geometric method formulated by this research can allow us to obtain new types of envelope surfaces [14].

Therefore, it is noted that the envelope of a planar cyclic transformation of a family **R** of corresponding conics is formed by a copy of conics tangent to each member of **R** in two corresponding points. It follows that for each pair of corresponding points of the envelope, only one conic passes; and that each point of the envelope has the same tangent to each conic of **R**. Similarly, the envelope of a cyclic transformation

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 4 (2021), pp. 323-332

of a family **S** of conjugated surfaces is a surface that touches each member of **S** in a curve. Each copy of envelope conics identifies infinite cyclids, such as **S**. Each point of the envelope has the same tangent plane to each member of **S**.

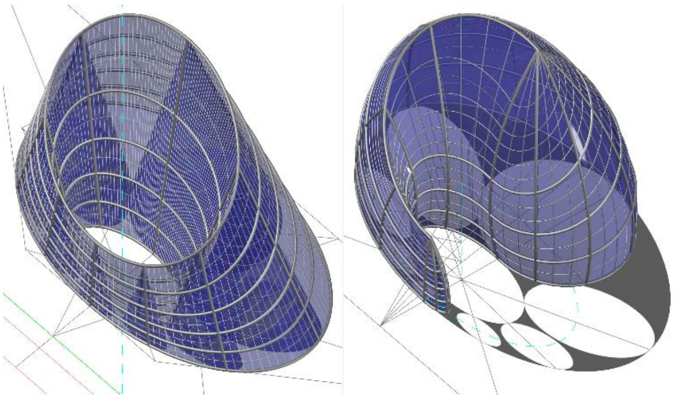


Figure 17: *on the left side there is a cichlid enveloped conjugated cone. On the right there is a cyclide that envelops surfaces with curved generatrices.*

Therefore, passing from a cyclic transformation between conics to that between surfaces we can identify different types of cyclids [15], in the condition in which the enveloped surfaces are conjugated, two by two to each other.

For example, the surface located in the left part of figure 17 there is a cyclide enveloping a sufficient number of cones which have, as their basis, the conics obtained in the first cyclic transformation (see figure 2).

The surfaces obtained with this procedure can be considered as a possible extension of Dupin's cyclids. The difference is that Dupin's cyclids are obtained, respectively, as enveloping surfaces of the cyclic transformations of three spheres. Instead, in our case, the surfaces can envelop, respectively, different types of conjugated surfaces. It follows that by determining the four cyclical transformations of three established generic conics, we can have numerous and interesting types of surfaces for architecture (fig. 17, 18).

As an alternative to assigning three coplanar conics that act as bases of the enveloped surfaces of a cyclide **k**, we can proceed with the sketch of a profile of the desired surface; to approximate it with conics obtained as projections of the same conic; and then to determine the various cyclical transformations between these conics. For example, in the attached case (fig.19), there are three cyclical transformations: the first is the cyclical defined by the conics $\mathbf{\Delta_1}$ and $\mathbf{\Delta_2}$ with the center in the point **U**; the second is between $\mathbf{\Delta_3}$ and $\mathbf{\Delta_4}$ with the center in point **V**, and the third is between $\mathbf{\Delta_1}$ and $\mathbf{\Delta_4}$, with the center in point **W**, and which acts as a connecting cyclic between the first two cyclics. The advantage of approximating free surfaces a priori using quadric cones (fig. 18) is to facilitate the process of constructive and, therefore, economic rationalization of the architectural project.

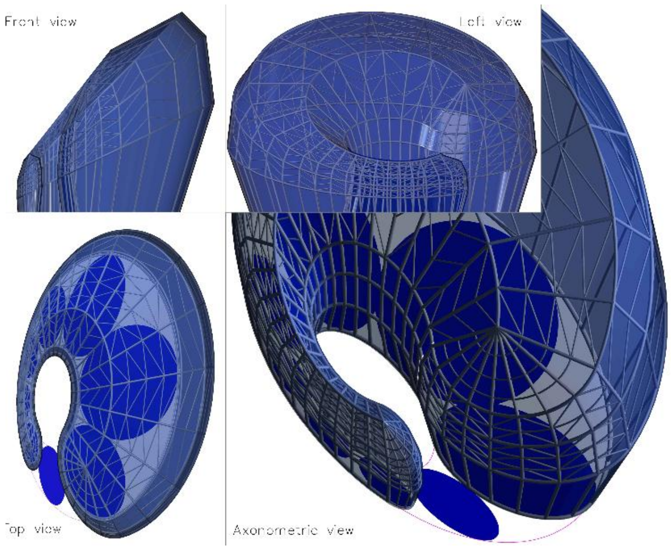


Figure 18*: the approximation of an elliptical cyclide using conical surfaces*

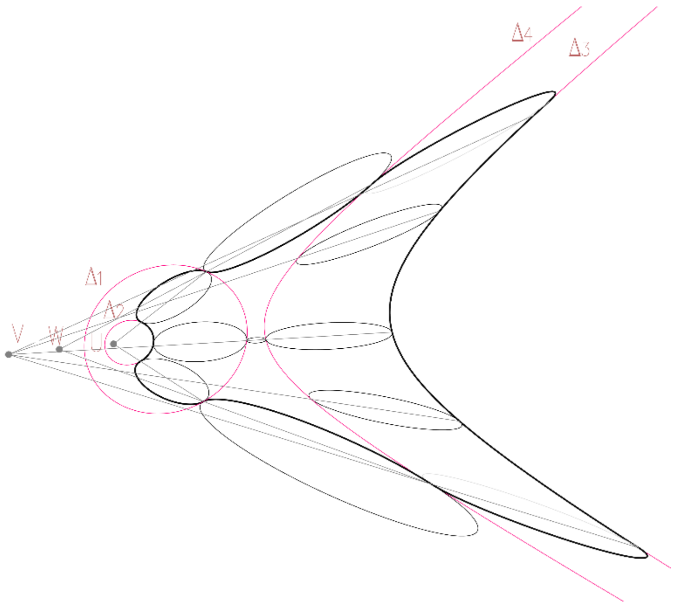


Figure 19: *an example of non-homothetic cyclic tangential connection.*

Conclusions

This research has generalized the problem of tangency between non-homothetic coplanar conics. We determined four copies of conics tangent to the established conics, which have been obtained as coplanar projections of the same conic. The points of tangency between the established conics and the determined ones are, respectively, the traces of the common generatrices of enveloped cones and those of enveloping ones. The vertices of the enveloped cones, in each case, belong to the same plane, which varies according to the combination of the three assigned cones (revise fig. 04). The points common to the three enveloped cones, in each case, are the vertices of the enveloping ones.

The methodology for the generation of the various cyclic transformations is mainly based on the computerized applications of the concepts of descriptive geometry. Such as: involution, intersections between conjugated cones, polarity in space, pole line, and polar cone. Like many geometric operations, even the cyclical transformations in the plane and space are identified by three enveloped conics or by two enveloping conics.

We studied the classifications of plane cyclic transformations between conics and spatial between cyclids, which are determined as the envelope of a successive series of cyclical transformations, conjugated, two by two. The centers of each of these transformations are the points of intersection of the polar lines of the corresponding plane with respect to a considered cone. We illustrated how to identify various types of cyclids by passing from a cyclic transformation between conics, to that between surfaces, which can be considered as a possible extension of Dupin's cyclids. The difference is that Dupin's cyclids are obtained, respectively, as enveloping surfaces of the cyclic transformations of three spheres. Instead, in our case, the surfaces can envelop, respectively, different types of conjugated surfaces. It follows that by determining the four cyclical transformations of three established coplanar conics, it is possible to have numerous and interesting surfaces for architecture (revise Fig.17,18).

This research offers the alternative to assigning three coplanar conics that act as bases of the enveloping surfaces, for example, by proceeding with a free sketch of the profile of the desired surface; to approximate it with conics obtained as projections of the same conic; and then to determine the various cyclical transformations between these conics. Therefore, the procedure allows us to obtain tangential continuity between conjugated cyclids, which are created as the envelope of a successive series of cyclic transformations of conics corresponding two by two, to each other.

Curves and surfaces created with this method can be considered as a priori approximation of free surfaces. The advantage of building cyclids through the use of conical surfaces, due to their property of being developable, allows to facilitate the process of constructive and, therefore, economic rationalization of the architectural project.

This research enriches the vocabulary of architectural forms with curves and surfaces that are more controllable as geometric locus.

Furthermore, this research has also didactic utility: in emphasizing the ability of descriptive geometry in solving complex spatial problems. Because the processes of creating models in space, through drawing, not only allow the visualization of perceived ideas but constantly reveal, in a simple and direct way, unexpected problems and unknown geometric properties. Hence, "descriptive geometry provides continuous examples of the passage from the known to the unknown".

It, therefore, provides architects and designers with alternative ways of designing envelope surfaces, with easily understandable geometric notions. The ability to translate the descriptive processes of this research into algorithms can help designers automate repetitive tasks.

References

[1] Apollonius of Perga, 200 BC, Conics, books V to VII : the arabic translation of the lost Greek original in the version of the Banu Musa. 1. Arabic version of the Conics (كتاب المخروطات) by (Ἀπολλώνιος; أبلونيوس).

[2] Given three geometric entities, each of which can be a point, a straight line or a circle, draw a circle (or circumferences) tangent all of them. There are a total of ten cases. The two easier ones include three points or three lines and the most difficult includes three circles. Cfr. Viete, F., 1600, *Apollonius Gallus seu, Exsuscitata Apollonii Pergaei peri epafwn Geometria*, Ad V. C. A. R. Paris.

[3] Van Romeen, A., 1596, *Problema Apolloniacum, Adrianum romanum constructum. Wirceburgi*, Typis Georgij Fleifchmanni., pp.18.

[4] Three cones are said to be conjugated to each other, when their coplanar sections are not only corresponding to each other two by two, but also have the centers of correspondence, both direct and inverse, belonging to the same line. That when the section. Cfr. Schiapparelli, G. V., 1862, "*Sulla trasformazione geometrica delle figure ed in particolare sulla trasformazione iperbolica*, Stamperia reale, Torino. pp. 47-51.

[5] Migliari, R., 2008, *il problema di Apollonio e la geometria descrittiva- The apollonian problem and descriptive geometry*, idee immagini- ideas image. pp. 22-33.

[6] The vertices of three established cones having a section in common can identify any position, and in the case in which it is horizontal, we will have improper points as the centers of the correspondences that exist, two by two, between the bases of these cones. Cfr. Schiapparelli (1862), op. cit.

[7] Enriques F., 1898, *Lezioni di geometria proiettiva*, Italian ed. pp. 208, 305.

[8] Fiedler W., 1873. *Trattato di geometria descrittiva*, Le Monnier, Firenze. p. 260.

[9] Bortolotti, E., 1942, *Geometra Descrittiva: lezioni redatte per uso degli studenti*, Cedam, Padova. p. 529.

[10] Pasi. C,. 1844, *Sunto di lezioni di Geometria descrittiva*, 2nd ed., Volume 1. Bizzoni. p. 65.

[11] De Lagrange. F. A., 1872, *Catalogue of a collection of models of ruled surfaces*, with an appendix, containing an account of the application of analysis to their investigation an d classification, by C. W. Merrifield, principal op the Printed by George E. Eyre and William Spottiswoode. London. p.29

[12] If in the plane of the base **Δv** of a quadric cone **v**, the line p is the polar of a point P with respect to **Δv**, the line that passes through the vertex **V** of v and through P is the pole line. The plane **γ** passing through **V** and **p** is the corresponding polar plane with respect to the cone **v**. Cfr. Fiedler (1873), Op. cit. pp. 215-216.

[13] Guglielmo, F., *1874. Trattato di geometria descrittiva,* Sucessori le Monnierp. pp. 253-254.

[14] The polar planes of the points of a straight line r not passing through the vertex V of a cone v, form a Sheaf whose axis r 'passes through V. for which r' is called the polar line of r. Hence the straight line r' contains the poles of r with respect to the conic sections with the planes for it. CFR. Guglielmo (1874) Op. Cit. p. 377

[15] Sereni, C., 1826. *Trattato di geometria descrittiva*, Stamperia di Filippo e Nicola De Romanis. Roma. (1826), parte III: *delle linee e delle superfici curve*. pp. 37- 40.

[16] Loria. G., 1885, *Ricerche intorno alla geometria della sfera, proprieta generali dello spazio di sfere,* In: Memorie della Reale accademia delle scienze di Torino. Italy, Stamperia reale. pp. 256-295.

[17] Severi, F., 1879-1961, *Geometria proiettiva,* University of Michigan Historical Math Collection.