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Galerkin-Vlasov Variational Method for the Elastic Buckling Analysis of SSCF and SSSS Rectangular Plates

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Abstract

This paper presents the Galerkin-Vlasov variational method for the elastic buckling analysis of SSCF and SSSS rectangular plates. The thin plate problems studied are: (i) simply supported along two opposite sides $x = 0$, and $x = a$, clamped along the third side $y = 0$, and free along the fourth side $y = b$; (ii) simply supported along the four sides $x = 0$, $x = a$, $y = 0$ and $y = b$. In each case the edges $x = 0$ and $x = a$ are subjected to uniform compressive load. Mathematically, the considered stability problem is a Boundary Value Problem (BVP) expressed by a domain fourth order partial differential equation (PDE) whose general solution should satisfy all the boundary conditions determined by the edge support conditions. By the Galerkin-Vlasov method, the unknown deflection shape function is chosen as the product of the eigenfunctions of a vibrating thin beam of identical span in the x direction and an unknown function of $y(G_n(y))$. The Galerkin-Vlasov variational integral equation is simplified using the Leibnitz rule, integration by parts and the orthogonality properties of the eigenfunctions of simply supported thin beams to a system of fourth order ordinary differential equations (ODEs). The general solution of the system of ODEs is obtained using trial function methods in terms of hyperbolic and trigonometric functions. The imposition of boundary conditions is used in each of the two cases to find the characteristic buckling equation. The buckling equation is obtained in each case as a transcendental equation, which is solved to obtain the eigenvalues from which the buckling loads are determined. The results obtained in each case for the buckling equation are identical to previous results obtained by other scholars who used classical methods and energy minimization methods. The results obtained for the buckling loads are in agreement with previously obtained solutions in the literature. The results obtained in each presented case in this study are found to be exact because exact shape functions were used in the x direction and the general solution was obtained for the domain PDE at every point in the plate domain. In addition, the solution obtained was made to satisfy all the boundary conditions at all the edges of the plate.

Keywords: Galerkin-Vlasov variational method, characteristic elastic buckling equation, critical elastic buckling load, elastic buckling problem, elastic buckling load coefficient.

I. INTRODUCTION

The elastic buckling analysis of plates subjected to various distributions of compressive loads applied in their plane is an important part of structural design of such members [1 – 10]. The plate may be assumed thin or thick, homogeneous or inhomogeneous; and the loads may vary uniformly or non-uniformly along the plate edges. The buckling problem may be elastic or inelastic. One of the first studies of plate buckling was conducted by Navier, who used the basic assumptions of Kirchhoff thin plate theory to formulate the stability equation of rectangular thin plates to include twisting. Saint Venant modified the Navier formulation to incorporate the edge axial and shear loads. Since the scholarly works of Navier and Saint Venant, several other studies have contributed significantly to the present day knowledge of buckling. Some of these works are presented by: Ullah et al [11, 12, 13, 14], Timoshenko [2], Timoshenko and Gere [3], Yu [8], Xiang et al [15], and Abodi [16]. Oguaghamba [17] and Oguaghamba et al [18] investigated the buckling and post-buckling loads of rectangular plates assumed to be thin and with material properties that are isotropic and homogeneous.

Various numerical and mathematical techniques have been used for the plate buckling problem. They include: Finite Difference Method (FDM), Differential Quadrature Method (DQM), Discrete Singular Convolution (DSC) Method, Finite Element Method (FEM), Ritz variational method, Finite Strip Method (FSM), Galerkin variational method, single finite Fourier sine transform method, double finite Fourier sine transform method and Galerkin-Kantorovich method.

Ibearugbulem [19] and Nwadike [20] studied the elastic buckling analysis of thin rectangular flat plates under uniform compressive loads using Ritz variational technique. They considered various boundary conditions and mostly presented one unknown displacement parameter solutions for the critical buckling loads. Osadebe et al [21] and Ibearugbulem et al [22] presented the elastic buckling analysis of SSSS plates by the Galerkin variational method that used the truncated Taylor-Maclaurin's series shape functions. Both studies considered thin rectangular isotropic homogeneous plates subjected to uniaxial uniform compressive load applied in-plane.

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Finite Difference method has been used for buckling problems by Abodi [16]. Boundary element method was used by Shi [5], and Shi and Bezine [6]. Finite Fourier sine integral transform methods were used for elastic buckling analysis of plates by Nwoji et al [23], and Onah et al [24].

In a recent study, Onyia et al [25] presented the Galerkin-Kantorovich method for solving the elastic stability problem of thin rectangular plate with two opposite simply supported sides and the remaining two edges clamped. Onyia et al [25] assumed the simply supported sides carry uniaxial uniform compressive load, and considered isotropic, homogeneous material properties. They obtained exact solutions for both the stability equation and the buckling loads. Other recent studies on plate buckling include: Lopatin and Morozov [26], Jafari and Azhari [27], Zureick [28], Seifi et al [29], Li et al [30], Wang et al [31], Mandal and Mishra [32] and Bouazza et al [33].

In the present study, the Galerkin-Vlasov method is presented for solving the elastic buckling problems of rectangular SSCF and SSSS plates subjected to uniform compressive loads acting on the simply supported edges.

II.THEORETICAL FRAMEWORK

The governing partial differential equation for the elastic buckling of rectangular thin plate (of dimensions $a \times b$) is generally given by:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = p(x, y) \quad (1)$$

The origin is assumed to be at a corner of the plate. Equation (1) is also expressed in more compact form as:

$$D\nabla^4 w + N_x w_{xx} + N_y w_{yy} + 2N_{xy} w_{xy} = p(x, y) \quad (2)$$

where $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

$$w_{xx} = \frac{\partial^2 w}{\partial x^2}, \; w_{yy} = \frac{\partial^2 w}{\partial y^2}, \; w_{xy} = \frac{\partial^2 w}{\partial x \partial y}$$

∇^4 is the biharmonic operator, $w(x, y)$ is the deflection, x, y are the in-plane Cartesian coordinates, N_x, N_y are the in-plane normal compressive loads on the edges, N_{xy} is the in-plane shear force D is the flexural rigidity of the plate.

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

E is the Young's modulus of elasticity, h is the plate thickness, μ is the Poisson's ratio of the plate material, $p(x, y)$ is the distributed transverse force on the plate domain.

The SSCF plate considered is shown in Figure 1.

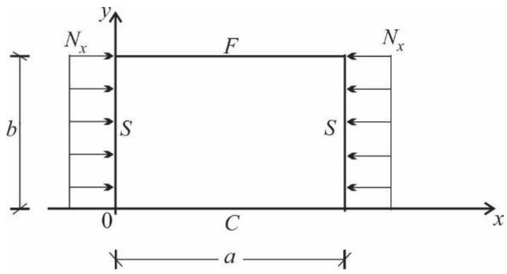

Figure 1: Rectangular thin plate simply supported on two opposite edges, clamped on the third edge and free on the fourth with the simply supported edges under uniform axial compressive load

The governing domain equation for the elastic buckling problem simplifies to:

$$D\nabla^4 w(x, y) + N_x w_{xx} = 0 \quad (3)$$

Since $N_y = 0$, $N_{xy} = 0$, $p(x, y) = 0$

$$\text{Thus, } \nabla^4 w(x, y) + \frac{N_x}{D}\frac{\partial^2 w(x, y)}{\partial x^2} = 0 \quad (4)$$

The boundary conditions are:

For the simply supported edges, $x = 0$, and $x = a$,

$$w(x = 0, y) = w(x = a, y) = 0 \quad (5)$$

$$w_{xx}(x = 0, y) = w_{xx}(x = a, y) = 0 \quad (6)$$

At the clamped edge, $y = 0$,

$$w(x, y = 0) = 0 \quad (7)$$

$$\frac{\partial w}{\partial y}(x, y = 0) = 0 \quad (8)$$

At the free edge, $y=b$,

$$M_{yy}(x, y = b) = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_{x, y=b} = 0 \quad (9)$$

$$V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right)\Bigg|_{x, y=b} = 0 \quad (10)$$

where M_{yy} is the bending moment, V_y is the shear force.

III. METHODOLOGY

By the Galerkin-Vlasov methodology, the shape function is chosen for the x-coordinate direction as the eigenfunctions of a vibrating thin beam of equivalent span and support conditions. Thus, the shape function in the x direction is assumed as Equation (11).

$$F_m(x) = \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \quad (11)$$

$m = 1, 2, 3, \ldots, \infty$

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We observe that $F_m(x)$ satisfies the Dirichlet boundary conditions at the simply supported edges. Hence it is an appropriate shape function for the considered problem. The deflection function is thus given by Equation (12) as follows:

$$w(x,y)=\sum_m^\infty\sum_n^\infty F_m(x)G_n(y)=\sum_m^\infty\sum_n^\infty G_n(y)\sin\frac{m\pi x}{a} \quad (12)$$

The Galerkin-Vlasov variational integral equation for the problem is given by Equation (13).

$$\int_0^b\int_0^a\left(\nabla^4\sum_m^\infty\sum_n^\infty G_n(y)\sin\frac{m\pi x}{a}+\frac{N_x}{D}\frac{\partial^2}{\partial x^2}\sum_m^\infty\sum_n^\infty G_n(y)\sin\frac{m\pi x}{a}\right)\sin\frac{\bar{m}\pi x}{a}dxdy=0 \quad (13)$$

where $\bar{m}$ assume integer values.

Explicitly, Equation (13) can be expressed as Equation (14).

$$\int_0^b\int_0^a\left(\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^2}{\partial y^4}\right)\sum_m^\infty\sum_n^\infty G_n(y)\sin\frac{m\pi x}{a}+\frac{N_x}{D}\frac{\partial^2}{\partial x^2}\sum_m^\infty\sum_n^\infty G_n(y)\sin\frac{m\pi x}{a}\right)\sin\frac{m\pi x}{a}dxdy=0 \quad (14)$$

Simplification of Equation (14) yields the integral equation as Equation (15).

$$\sum_m^\infty\sum_n^\infty\int_0^b\int_0^a\left\{\left(\left(\frac{m\pi}{a}\right)^4G_n(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+G_n^{iv}(y)\right)\sin\frac{m\pi x}{a}-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}G_n(y)\sin\frac{m\pi x}{a}\right\}\sin\frac{m\pi x}{a}dxdy=0 \quad (15)$$

Further simplification of Equation (15) gives the integral equation as Equation (16).

$$\sum_m^\infty\sum_n^\infty\int_0^b\int_0^a\left\{\left(\left(\frac{m\pi}{a}\right)^4G_n(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+G_n^{iv}(y)\right)-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}G_n(y)\right\}\sin\frac{m\pi x}{a}\sin\frac{\bar{m}\pi x}{a}dxdy=0 \quad (16)$$

Simplifying Equation (16) further yields the Galerkin-Vlasov integral equation for the considered problem as Equation (17). Thus,

$$\sum_m^\infty\sum_n^\infty\int_0^b\left\{G_n^{iv}(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+\left(\left(\frac{m\pi}{a}\right)^4-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)G_n(y)\right\}dy\int_0^a\sin\frac{m\pi x}{a}\sin\frac{\bar{m}\pi x}{a}dx=0 \quad (17)$$

Simplifying again, Equation (17) becomes Equation (18).

$$\sum_m^\infty\sum_n^\infty\int_0^b\left\{G_n^{iv}(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+\left(\left(\frac{m\pi}{a}\right)^4-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)G_n(y)\right\}dy\cdot I_1=0 \quad (18)$$

in which,

$$I_1=\int_0^a\sin\frac{m\pi x}{a}\sin\frac{\bar{m}\pi x}{a}dx \quad (19)$$

The orthogonality properties of the sinusoidal functions in the integrand forI_1 are used to obtain the following results for the integration problem described by Equation (19):

For $m\neq\bar{m},\ I_1=0$
$m=\bar{m},\ I_1\neq 0$

Hence, Equation (18) simplifies to Equation (20).

$$\sum_m^\infty\sum_n^\infty\int_0^b\left(G_n^{iv}(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+\left(\left(\frac{m\pi}{a}\right)^4-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)G_n(y)\right)dy=0 \quad (20)$$

This Equation (20) is true if the integrand vanishes for all, m, n. This leads to the system of homogeneous fourth order ordinary differential equations (ODEs) in $G_n(y)$ expressed by Equation (21).

$$G_n^{iv}(y)-2\left(\frac{m\pi}{a}\right)^2G_n''(y)+\left(\left(\frac{m\pi}{a}\right)^4-\left(\frac{m\pi}{a}\right)^2\frac{N_x}{D}\right)G_n(y)=0 \quad (21)$$

Let $\alpha_m=\frac{m\pi}{a}$ (22)

Then, Equation (21) is expressed in terms of α_m as Equation (23).

$$G_n^{iv}(y)-2\alpha_m^2G_n''(y)+\left(\alpha_m^4-\alpha_m^2\frac{N_x}{D}\right)G_n(y)=0 \quad (23)$$

IV. GENERAL SOLUTION FOR $G_n(y)$

The general solution for $G_n(y)$ is assumed in exponential form as Equation (24):

$$G_n(y)=A\exp\Omega_n y \quad (24)$$

where A and Ω_n are parameters to be found.

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Then, the fourth order ODE becomes the algebraic problem given by Equation (25).

$$\left(\Omega_n^4 - 2\alpha_m^2\Omega_n^2 + \left(\alpha_m^4 - \alpha_m^2 \frac{N_x}{D}\right)\right) A \exp \Omega_n y = 0 \qquad (25)$$

For nontrivial solutions, A exp $\Omega_n y$ is not equal to 0, and the auxillary equation results from Equation (25) as Equation (26).

$$\Omega_n^4 - 2\alpha_m^2\Omega_n^2 + \alpha_m^4 - \alpha_m^2 \frac{N_x}{D} = 0 \qquad (26)$$

Solving Equation (26), we obtain by completing the squares:

$$(\Omega_n^2 - \alpha_m^2)^2 = \alpha_m^2 \frac{N_x}{D} \qquad (27)$$

Taking the square root of both sides of the Equation (27), we obtain Equation (28).

$$\Omega_n^2 - \alpha_m^2 = \sqrt{\alpha_m^2 \frac{N_x}{D}} \qquad (28)$$

Hence, solving for Ω_n^2, we obtain:

$$\Omega_n^2 = \alpha_m^2 \pm \sqrt{\alpha_m^2 \frac{N_x}{D}} \qquad (29)$$

Hence the two possibilities for Ω_n are:

$$\Omega_n^2 = \alpha_m^2 + \sqrt{\alpha_m^2 \frac{N_x}{D}} \qquad (30)$$

Or, $\Omega_n^2 = \alpha_m^2 - \sqrt{\alpha_m^2 \frac{N_x}{D}} = -\left(-\alpha_m^2 + \sqrt{\alpha_m^2 \frac{N_x}{D}}\right) \qquad (31)$

The four roots are thus,

$$\Omega_n = \left(\alpha_m^2 + \sqrt{\alpha_m^2 \frac{N_x}{D}}\right)^{1/2} = \beta_{1m} \qquad (32)$$

$$\Omega_n = \pm i\left(\sqrt{\alpha_m^2 \frac{N_x}{D}} - \alpha_m^2\right)^{1/2} = \pm i\beta_{2m} \qquad (33)$$

where $\beta_{1m} = \left(\sqrt{\alpha_m^2 \frac{N_x}{D}} + \alpha_m^2\right)^{1/2} \qquad (34)$

$$\beta_{2m} = \left(\sqrt{\alpha_m^2 \frac{N_x}{D}} - \alpha_m^2\right)^{1/2} \qquad (35)$$

The general solution for $G_n(y)$ is:

$$G_n(y) = c_{1m} \cosh \beta_{1m} y + c_{2m} \sinh \beta_{1m} y + c_{3m} \cos \beta_{2m} y + c_{4m} \sin \beta_{2m} y \qquad (36)$$

where c_{1m}, c_{2m}, c_{3m} and c_{4m} are the integration constants.

IV.I Imposition of boundary conditions on $w(x, y)$ along the edges $y = 0$ and $y = b$

Case of SSCF plate

Using Equations (12) and (36), the general solution for $w(x, y)$ is found as the expression given by Equation (37).

$$w(x, y) = \sum_m^\infty \left(c_{1m} \cosh \beta_{1m} y + c_{2m} \sinh \beta_{1m} y + c_{3m} \cos \beta_{2m} y + c_{4m} \sin \beta_{2m} y\right) \sin \frac{m\pi x}{a} \qquad (37)$$

The boundary condition for $w(x, y)$ at the edge $y = 0$ is given by:

$$w(x, y = 0) = \sum_{m=1}^\infty G_n(y) \sin \frac{m\pi x}{a}\bigg|_{x, y=0} = 0 \qquad (38)$$

Hence, the boundary condition Equation (38) simplifies to:

$$\therefore G_n(y = 0) = 0 \qquad (39)$$

Similarly, the boundary condition on $\frac{\partial w}{\partial y}$ at $y = 0$ is given by:

$$\frac{\partial w}{\partial y}(x, y = 0) = \frac{\partial}{\partial y} \sum G_n(y) \sin \frac{m\pi x}{a} = 0 \qquad (40)$$

Hence,

$$G_n'(y = 0) = 0 \qquad (41)$$

The boundary condition at the free edge $y = b$ is:

$$M_{yy}(x, y = b) = -D\left(G_n''(y) \sin \frac{m\pi x}{a} - \left(\frac{m\pi}{a}\right)^2 \mu G_n(y) \sin \frac{m\pi x}{a}\right) = 0 \qquad (42)$$

Hence,

$$\therefore \left(G_n''(y) - \mu\left(\frac{m\pi}{a}\right)^2 G_n(y)\right)\bigg|_{y=b} \sin \frac{m\pi x}{a} = 0 \qquad (43)$$

Or, $\left(G_n''(y = b) - \mu\alpha_m^2 G_n(y = b)\right) = 0 \qquad (44)$

The boundary condition on V_y at the free edge $y = b$ is:

$$V_y\big|_{y=b} = 0 = -D\left(G_n'''(y) \sin \frac{m\pi x}{a} - (2 - \mu)\alpha_m^2 G_n'(y) \sin \frac{m\pi x}{a}\right) = 0 \qquad (45)$$

Hence, Equation (45) becomes:

$$\therefore G_n'''(y = b) - (2 - \mu)\alpha_m^2 G_n'(y = b) = 0 \qquad (46)$$

Taking the derivatives of $G_n(y)$ with respect to y, we have:

$$G_n'(y) = c_{1m}\beta_{1m} \sinh \beta_{1m} y + c_{2m}\beta_{1m} \cosh \beta_{1m} y - c_{3m}\beta_{2m} \sin \beta_{2m} y + c_{4m}\beta_{2m} \cos \beta_{2m} y \qquad (47)$$

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$$G_n''(y) = c_{1m}\beta_{1m}^2 \cosh\beta_{1m}y + c_{2m}\beta_{1m}^2 \sinh\beta_{1m}y - c_{3m}\beta_{2m}^2 \cos\beta_{2m}y - c_{4m}\beta_{2m}^2 \sin\beta_{2m}y \quad (48)$$

$$G_n'''(y) = c_{1m}\beta_{1m}^3 \sinh\beta_{1m}y + c_{2m}\beta_{1m}^3 \cosh\beta_{1m}y + c_{3m}\beta_{2m}^3 \sin\beta_{2m}y - c_{4m}\beta_{2m}^3 \cos\beta_{2m}y \quad (49)$$

Hence from Equation (39), we have:

$$G_n(y=0) = c_{1m} + c_{3m} = 0 \quad (50)$$

So, $c_{3m} = -c_{1m}$ (51)

From Equation (41), we have:

$$G_n'(y=0) = c_{2m}\beta_{1m} + c_{4m}\beta_{2m} = 0 \quad (52)$$

Hence, $c_{4m} = -\dfrac{c_{2m}\beta_{1m}}{\beta_{2m}}$ (53)

From Equation (44), we obtain

$$c_{1m}\beta_{1m}^2 \cosh\beta_{1m}b + c_{2m}\beta_{1m}^2 \sinh\beta_{1m}b - c_{3m}\beta_{2m}^2 \cos\beta_{2m}b - c_{4m}\beta_{2m}^2 \sin\beta_{2m}b - \mu\alpha_m^2(c_{1m}\cosh\beta_{1m}b + c_{2m}\sinh\beta_{1m}b + c_{3m}\cos\beta_{2m}b + c_{4m}\sin\beta_{2m}b) = 0 \quad (54)$$

Simplifying, Equation (54) gives Equation (55):

$$c_{1m}(\beta_{1m}^2 - \mu\alpha_m^2)\cosh\beta_{1m}b + c_{2m}(\beta_{2m}^2 - \mu\alpha_m^2)\sinh\beta_{1m}b - c_{3m}(\beta_{2m}^2 + \mu\alpha_m^2)\cos\beta_{2m}b - c_{4m}(\beta_{2m}^2 + \mu\alpha_m^2)\sin\beta_{2m}b = 0 \quad (55)$$

Further simplification, of Equation (55) using Equations (51) and (53) gives Equation (56).

$$c_{1m}(\beta_{1m}^2 - \mu\alpha_m^2)\cosh\beta_{1m}b + c_{2m}(\beta_{1m}^2 - \mu\alpha_m^2)\sinh\beta_{1m}b + c_{1m}(\beta_{2m}^2 + \mu\alpha_m^2)\cos\beta_{2m}b + \frac{c_{2m}\beta_{1m}}{\beta_{2m}}(\beta_{2m}^2 + \mu\alpha_m^2)\sin\beta_{2m}b = 0 \quad (56)$$

Using Equation (46) we obtain:

$$c_{1m}\beta_{1m}^3 \sinh\beta_{1m}b + c_{2m}\beta_{1m}^3 \cosh\beta_{1m}b + c_{3m}\beta_{2m}^3 \sin\beta_{2m}b - c_{4m}\beta_{2m}^3 \cos\beta_{2m}b - (2-\mu)\alpha_m^2(c_{1m}\beta_{1m}\sinh\beta_{1m}b + c_{2m}\beta_{1m}\cosh\beta_{1m}b - c_{3m}\beta_{2m}\sin\beta_{2m}b + c_{4m}\beta_{2m}\cos\beta_{2m}b) = 0 \quad (57)$$

Simplifying, Equation (57) gives Equation (58):

$$c_{1m}(\beta_{1m}^3 - (2-\mu)\alpha_m^2\beta_{1m})\sinh\beta_{1m}b + c_{2m}(\beta_{1m}^3 - (2-\mu)\alpha_m^2\beta_{1m})\cosh\beta_{1m}b + c_{3m}(\beta_{2m}^3 + (2-\mu)\alpha_m^2\beta_{2m})\sin\beta_{2m}b - c_{4m}(\beta_{2m}^3 + (2-\mu)\alpha_m^2\beta_{2m})\cos\beta_{2m}b = 0 \quad (58)$$

Further simplification of Equation (58) gives Equation (59) as follows:

$$c_{1m}\beta_{1m}(\beta_{1m}^2 - (2-\mu)\alpha_m^2)\sinh\beta_{1m}b + c_{2m}\beta_{1m}(\beta_{1m}^2 - (2-\mu)\alpha_m^2)\cosh\beta_{1m}b + c_{3m}\beta_{2m}(\beta_{2m}^2 + (2-\mu)\alpha_m^2)\sin\beta_{2m}b - c_{4m}\beta_{2m}(\beta_{2m}^2 + (2-\mu)\alpha_m^2)\cos\beta_{2m}b = 0 \quad (59)$$

It can be shown that Equation (60) is true.

$$\beta_{1m}^2 - \mu\alpha_m^2 = \beta_{2m}^2 + (2-\mu)\alpha_m^2 \quad (60)$$

From Equation (60), we have

$$\beta_{2m}^2 - \beta_{1m}^2 = -\mu\alpha_m^2 - (2-\mu)\alpha_m^2 = -2\alpha_m^2 \quad (61)$$

Also from the expressions for β_{1m} and β_{2m}, (Equations (34) and (35)) we have:

$$\beta_{2m}^2 - \beta_{1m}^2 = \left(\sqrt{\alpha_m^2 \frac{N_x}{D}} - \alpha_m^2\right) - \left(\sqrt{\alpha_m^2 \frac{N_x}{D}} + \alpha_m^2\right) = -2\alpha_m^2 \quad (62)$$

Hence, $\beta_{1m}^2 - \mu\alpha_m^2 = \beta_{2m}^2 + (2-\mu)\alpha_m^2 = \lambda_1$ (63)

Similarly,

$$\beta_{2m}^2 + \mu\alpha_m^2 = \beta_{1m}^2 - (2-\mu)\alpha_m^2 = \lambda_2 \quad (64)$$

Then, the boundary conditions become:

$$c_{1m}(\lambda_1\cosh\beta_{1m}b + \lambda_2\cos\beta_{2m}b) + c_{2m}\left(\lambda_1\sinh\beta_{1m}b + \frac{\beta_{1m}}{\beta_{2m}}\lambda_2\sin\beta_{2m}b\right) = 0 \quad (65)$$

and,

$$c_{1m}(\beta_{1m}\lambda_2\sinh\beta_{1m}b - \beta_{2m}\lambda_1\sin\beta_{2m}b) + c_{2m}(\beta_{1m}\lambda_2\cosh\beta_{1m}b + \beta_{1m}\lambda_1\cos\beta_{2m}b) = 0 \quad (66)$$

As a matrix, the system of Equations (65) and (66) becomes:

$$\begin{pmatrix} (\lambda_1\cosh\beta_{1m}b + \lambda_2\cos\beta_{2m}b) & \left(\lambda_1\sinh\beta_{1m}b + \frac{\beta_{1m}}{\beta_{2m}}\lambda_2\sin\beta_{2m}b\right) \\ (\beta_{1m}\lambda_2\sinh\beta_{1m}b - \beta_{2m}\lambda_1\sin\beta_{2m}b) & (\beta_{1m}\lambda_2\cosh\beta_{1m}b + \beta_{1m}\lambda_1\cos\beta_{2m}b) \end{pmatrix}\begin{pmatrix} c_{1m} \\ c_{2m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

…(67)

The stability equation is obtained for nontrivial solutions as the equation:

$$\begin{vmatrix} (\lambda_1\cosh\beta_{1m}b + \lambda_2\cos\beta_{2m}b) & \left(\lambda_1\sinh\beta_{1m}b + \frac{\beta_{1m}}{\beta_{2m}}\lambda_2\sin\beta_{2m}b\right) \\ (\beta_{1m}\lambda_2\sinh\beta_{1m}b - \beta_{2m}\lambda_1\sin\beta_{2m}b) & (\beta_{1m}\lambda_2\cosh\beta_{1m}b + \beta_{1m}\lambda_1\cos\beta_{2m}b) \end{vmatrix} = 0$$

…(68)

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Expansion of the determinant yields:

$$(\lambda_1 \cosh\beta_{1m}b + \lambda_2 \cos\beta_{2m}b)(\beta_{1m}\lambda_2 \cosh\beta_{1m}b + \beta_{1m}\lambda_1 \cos\beta_{2m}b) - \left(\lambda_1 \sinh\beta_{1m}b + \frac{\beta_{1m}}{\beta_{2m}}\lambda_1 \sin\beta_{2m}b\right)\times (\beta_{1m}\lambda_2 \sinh\beta_{1m}b - \beta_{2m}\lambda_1 \sin\beta_{2m}b) = 0 \quad (69)$$

Hence, simplification of Equation (69) gives Equation (70).

$$\beta_{1m}\lambda_1\lambda_2 \cosh^2\beta_{1m}b + \lambda_2^2\beta_{1m}\cosh\beta_{1m}b\cos\beta_{2m}b + \lambda_1^2\beta_{1m}\cosh\beta_{1m}b\cos\beta_{2m}b + \lambda_1\lambda_2\beta_{1m}\cos^2\beta_{2m}b - \left(\beta_{1m}\lambda_1\lambda_2\sinh^2\beta_{1m}b + \frac{\beta_{1m}}{\beta_{2m}}\lambda_2\beta_{1m}\lambda_2\sinh\beta_{1m}b\sin\beta_{2m}b - -\lambda_1^2\beta_{2m}\sinh\beta_{1m}b\sin\beta_{2m}b - \frac{\beta_{1m}^2}{\beta_{2m}}\lambda_2\lambda_1\beta_{2m}\sin^2\beta_{2m}b\right) = 0 \quad \ldots(70)$$

Simplification of Equation (70) gives:

$$\lambda_1\lambda_2\beta_{1m}(\cosh^2\beta_{1m}b - \sinh^2\beta_{1m}b) + \lambda_1\lambda_2\beta_{1m}(\cos^2\beta_{2m}b + \sin^2\beta_{2m}b) + (\lambda_2^2\beta_{1m} + \lambda_1^2\beta_{1m})\cosh\beta_{1m}b\cos\beta_{2m}b - \left(\frac{\lambda_2^2\beta_{1m}^2}{\beta_{2m}} - \lambda_1^2\beta_{2m}\right)\sinh\beta_{1m}b\sin\beta_{2m}b = 0 \quad (71)$$

From trigonometric and hyperbolic identities, Equation (71) can be simplified using Equations (72) and (73)

$$\cosh^2\beta_{1m}b - \sinh^2\beta_{1m}b = 1 \quad (72)$$

$$\cos^2\beta_{2m}b + \sin^2\beta_{2m}b = 1 \quad (73)$$

Hence, the simplification of Equation (71) using the trigonometric and hyperbolic identities gives Equation (74).

$$2\lambda_1\lambda_2\beta_{1m} + (\lambda_2^2 + \lambda_1^2)\beta_{1m}\cosh\beta_{1m}b\cos\beta_{2m}b - \left(\frac{\lambda_2^2\beta_{1m}^2}{\beta_{2m}} - \lambda_1^2\beta_{2m}\right)\sinh\beta_{1m}b\sin\beta_{2m}b = 0 \quad (74)$$

Division of Equation (74) by β_{1m}, gives Equation (75).

$$2\lambda_1\lambda_2 + (\lambda_2^2 + \lambda_1^2)\cosh\beta_{1m}b\cos\beta_{2m}b - \frac{1}{\beta_{1m}}\left(\frac{\lambda_2^2\beta_{1m}^2}{\beta_{2m}} - \lambda_1^2\beta_{2m}\right)\sinh\beta_{1m}b\sin\beta_{2m}b = 0 \quad (75)$$

Further simplification of Equation (75) yields Equation (76).

$$2\lambda_1\lambda_2 + (\lambda_1^2 + \lambda_2^2)\cosh\beta_{1m}b\cos\beta_{2m}b - \frac{1}{\beta_{1m}\beta_{2m}}\left(\lambda_2^2\beta_{1m}^2 - \lambda_1^2\beta_{2m}^2\right)\sinh\beta_{1m}b\sin\beta_{2m}b = 0 \quad (76)$$

The characteristic buckling equation – Equation (76) – which has been derived and observed to be a transcendental equation is solved by computational software tools to find the eigenvalues (roots). The obtained eigenvalues are used to find the buckling loads for different varying plate aspect ratios and for a Poisson's ratio $\mu = 0.25$. The critical buckling load coefficients $K(a/b)$ of SSCF plates obtained in this study using the Galerkin-Vlasov method for $\mu = 0.25$, and varying aspect ratios are presented in Table 1 for uniform compressive load applied at the simply supported edges $x = 0$, and $x = a$.

Table 1. Critical buckling load factors (coefficients) $K(a/b)$ for SSCF rectangular thin plates under uniform axial compression

$$N_{x_{cr}} = K\left(a/b\right)\frac{D\pi^2}{b^2}$$

Aspect ratio, a/b	Corresponding buckling mode, m	Present study $K(a/b)$	Wang et al [1] $K(a/b)$
0.5	$m = 1$	4.518	4.518
1.0	$m = 1$	1.698	1.698
1.5	$m = 1$	1.339	1.339
2.0	$m = 1$	1.386	1.386
2.5	$m = 2$	1.432	1.432
3.0	$m = 2$	1.339	1.339
3.5	$m = 2$	1.336	1.336
4.0	$m = 3$	1.386	1.386
4.5	$m = 3$	1.339	1.339
5.0	$m = 3$	1.329	1.329
5.5	$m = 3$	1.347	1.347
6.0	$m = 3$	1.339	1.339

IV.II Imposition of boundary conditions along $y = 0$, and $y = b$ for case of SSSS plates

For simply supported plates, the boundary conditions along the $y = 0$, and $y = b$ edges are given by:

$$w(x, y = 0) = 0$$

$$\frac{\partial^2 w}{\partial y^2}(x, y = 0) = 0$$

$$w(x, y = b) = 0 \quad (77)$$

$$\frac{\partial^2 w}{\partial y^2}(x, y = b) = 0$$

These boundary conditions imply that:

$$G_n(y = 0) = 0$$

$$G_n''(y = 0) = 0$$

$$G_n(y = b) = 0 \quad (78)$$

$$G_n''(y = b) = 0$$

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Thus,

$$G_n'(y=0) = c_{1m} + c_{3m} = 0 \quad (79)$$

$$G_n''(y=0) = c_{1m}\beta_{1m}^2 - c_{3m}\beta_{2m}^2 = 0 \quad (80)$$

$$G_n(y=b) = c_{1m}\cosh\beta_{1m}b + c_{2m}\sinh\beta_{1m}b + c_{3m}\cos\beta_{2m}b + c_{4m}\sin\beta_{2m}b = 0 \quad (81)$$

$$G_n''(y=b) = c_{1m}\beta_{1m}^2\cosh\beta_{1m}b + c_{2m}\beta_{1m}^2\sinh\beta_{1m}b - c_{3m}\beta_{2m}^2\cos\beta_{2m}b - c_{4m}\beta_{2m}^2\sin\beta_{2m}b = 0 \quad (82)$$

Solving Equations (79) and (80) simultaneously, we have:

$$c_{1m} = c_{3m} = 0 \quad (83)$$

Then the system of homogeneous equations become

$$c_{2m}\sinh\beta_{1m}b + c_{4m}\sin\beta_{2m}b = 0 \quad (84)$$

$$c_{2m}\beta_{1m}^2\sinh\beta_{1m}b - c_{4m}\beta_{2m}^2\sin\beta_{2m}b = 0 \quad (85)$$

As a matrix, Equations (84) and (85) become:

$$\begin{pmatrix} \sinh\beta_{1m}b & \sin\beta_{2m}b \\ \beta_{1m}^2\sinh\beta_{1m}b & -\beta_{2m}^2\sin\beta_{2m}b \end{pmatrix}\begin{pmatrix} c_{2m} \\ c_{4m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (86)$$

The determinant of the coefficient matrix is required to vanish for nontrivial solutions. Hence the elastic buckling equation is given by Equation (87).

$$\begin{vmatrix} \sinh\beta_{1m}b & \sin\beta_{2m}b \\ \beta_{1m}^2\sinh\beta_{1m}b & -\beta_{2m}^2\sin\beta_{2m}b \end{vmatrix} = 0 \quad (87)$$

Expansion of the determinant in Equation (87) yields:

$$-\beta_{2m}^2\sin\beta_{2m}b\sinh\beta_{1m}b - \beta_{1m}^2\sinh\beta_{1m}b\sin\beta_{2m}b = 0 \quad (88)$$

Simplifying,

$$-(\beta_{1m}^2 + \beta_{2m}^2)\sin\beta_{2m}b\sinh\beta_{1m}b = 0 \quad (89)$$

For nontrivial solutions,

$$\sin\beta_{2m}b = 0 \quad (90)$$

$$\beta_{2m}b = \sin^{-1}0 = n\pi, \qquad n = 1, 2, 3, \ldots \quad (91)$$

$$\beta_{2m} = \frac{n\pi}{b} \quad (92)$$

From Equation (35), we have

$$\beta_{2m} = \left(\sqrt{\alpha_m^2\frac{N_x}{D}} - \alpha_m^2\right)^{1/2} = \frac{n\pi}{b} \quad (93)$$

Squaring both sides of Equation (93) gives Equation (94) as follows:

$$\sqrt{\alpha_m^2\frac{N_x}{D}} - \alpha_m^2 = \left(\frac{n\pi}{b}\right)^2 \quad (94)$$

Hence, simplifying Equation (94) gives:

$$\sqrt{\alpha_m^2\frac{N_x}{D}} = \alpha_m^2 + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad (95)$$

Hence, squaring both sides of Equation (95) gives:

$$\alpha_m^2\frac{N_x}{D} = \left(\frac{m\pi}{a}\right)^2\frac{N_x}{D} = \left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 \quad (96)$$

Then, Equation (96) is expressed such that N_x is the subject as follows:

$$N_x = D\left(\frac{a}{m\pi}\right)^2\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)^2 \quad (97)$$

Simplifying Equation (97) gives:

$$N_x = D\frac{a^2}{(m\pi)^2}\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) \quad (98)$$

Further simplification gives:

$$N_x = D\left(\left(\frac{m\pi}{a}\right)^2 + 2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{a}{(m\pi)}\right)^2\left(\frac{n\pi}{b}\right)^4\right) \quad (99)$$

Further simplification of Equation (99) gives Equation (100) as follows:

$$N_x = D\left(\left(\frac{m\pi}{a}\right)^2 + 2\left(\frac{n\pi}{b}\right)^2 + \frac{a^2}{m^2}\frac{n^4\pi^2}{b^4}\right) \quad (100)$$

Let $a/b = r$, then $a = br$, and Equation (100) becomes in terms of r and b;

$$N_x = D\pi^2\left(\frac{m^2}{b^2r^2} + \frac{2n^2}{b^2} + \frac{r^2}{m^2}\frac{n^4}{b^2}\right) \quad (101)$$

Simplifying Equation (101) gives:

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{m^2}{r^2} + 2n^2 + \frac{r^2n^4}{m^2}\right) \quad (102)$$

Equation (102) is expressed in terms of buckling coefficients, $K(r, m, n)$ as:

$$N_x = K(r, m, n)\frac{D\pi^2}{b^2} \quad (103)$$

where, $$K(r, m, n) = \left(\frac{m^2}{r^2} + 2n^2 + \frac{r^2n^4}{m^2}\right) \quad (104)$$

For $m = n = 1$,

$$K(r, m = n = 1) = \left(\frac{1}{r^2} + 2 + r^2\right) = K\left(a/b\right) \quad (105)$$

$K(r)$ is evaluated for SSSS plates for different values of the aspect ratio and presented in Table 2 which also shows the agreement of the obtained results with results by Iyengar [4] and Nwoji et al [23].

Table 2. Elastic buckling load coefficients $K(a/b)$ for simply supported rectangular thin plate under uniaxial uniform compressive load applied at the two opposite simply supported edges for varying values of the aspect ratio, r.

$r = a/b$	Present study $K(a/b)$	Iyengar [4] $K(a/b)$	Nwoji et al [23] $K(a/b)$
0.1	102.01	102.01	102.01
0.2	27.04	27.04	27.04
0.3	13.201111	13.2011	13.2011
0.4	8.41	8.41	8.41
0.5	6.25	6.25	6.25
0.6	5.137778	5.1378	5.1378
0.7	4.530816	4.5308	4.5308
0.8	4.2025	4.2025	4.2025
0.9	4.044568	4.0446	4.0446
1.0	4	4	4

V. DISCUSSION

This study has presented the Galerkin-Vlasov method for the elastic buckling analysis of SSCF and SSSS rectangular thin plates subjected to uniform axial compression at the two opposite simply supported edges ($x = 0$, and $x = a$). The governing domain equation is a simplification of the general domain equation for thin plate buckling under a general state of in-plane loads and distributed transverse loads given by Equation (1). The governing domain equation is obtained from Equation (1) for the case when $N_y = 0$, $N_{xy} = 0$, and $p(x, y) = 0$, and is given by Equation (4). The boundary conditions for SSCF plates are given by Equations (5 – 10). Using the Galerkin-Vlasov methodology, the shape function in the x coordinate direction $F_m(x)$ is chosen as the eigenfunctions of a vibrating Euler-Bernoulli beam of equivalent span and support conditions, as Equation (11). The deflection function used is thus given as the product of unknown function in the y-coordinate $G_n(y)$ and $F_m(x)$ as Equation (12). The Galerkin-Vlasov variational integral equation for the considered problem for both SSCF and SSSS plates was thus found as Equation (13) expressed explicitly as Equation (14). Simplification of the Galerkin-Vlasov integral equation resulted in Equation (18) which further reduced to Equation (20). Ultimately the condition for the validity of equation (20) was found to be the vanishing of the integrand, resulting in the system of ODEs in $G_n(y)$ given by Equation (21), or more compactly as Equation (23). A trial function solution assuming the exponential trial function in Equation (24) led to the algebraic problem in Equation (25). The condition for nontrivial solutions of Equation (25) resulted in the homogeneous algebraic equation – Equation (26). The four roots of Equation (26) were found as Equations (32) and (33). The four roots were used to establish the solution basis of linearly independent solutions resulting in the general equation presented in Equation (36) which contains integration constants c_{1m}, c_{2m}, c_{3m} and c_{4m}. The general solution for $w(x, y)$ was thus obtained as Equation (37).

V.I Discussion on SSCF plates

The boundary conditions equations for SSCF plates were expressed in terms of $G_n(y)$ and its derivatives as Equations (39), (41), (44) and (46). The boundary conditions Equations (39), (41), (44) and (46) were used to set up a set of four homogeneous equations – Equations (50), (52), (56) and (59). The set of homogeneous equations was further reduced with the aid of relations between two constants given by Equations (51) and (53). The resulting simplification is a set of two homogeneous equations expressed by Equations (65) and (66) and presented as the matrix in Equation (67). The stability equation, obtained from the condition for nontrivial solutions was found as Equation (68). The simplification and use of trigonometric and hyperbolic identities of mathematics resulted in the transcendental equation given as Equation (76). Transcendental equations are usually difficult to solve in closed form. The eigenvalues of the transcendental equation were found using computational software tools that deploy iteration, and the buckling loads obtained from the eigenvalues. The buckling loads computed for various buckling modes and varying aspect ratios, and for Poisson's ratio $\mu = 0.25$ are presented in Table 1. Table 1 shows that the present results are identical with previous results obtained by Wang et al [1] thus validating the present study.

V.II Discussion on simply supported plates

For the case of SSSS plates considered, the boundary conditions for the edges $y = 0$ and $y = b$, expressed as the system of four equations in Equation (77) are given as the system of equations in terms of $G_n(y)$ and its second derivative with respect to y as Equation (78). Explicitly, the imposition of boundary conditions results in the system of four equations – Equations (79), (80), (81) and (82). Equations (79) and (80) are solved simultaneously yielding the solutions for two constants as Equation (83). The vanishing of two integration constants as expressed by Equation (83) reduce the resulting equations to a set of two equations – Equations (84) and (85). The equations are expressed as the matrix equation in Equation (86). The condition for nontrivial solutions requires the vanishing of the determinant of the coefficient matrix yielding the equation given by Equation (87). Expansion of the determinant and further simplification gave the elastic buckling equation as the transcendental equation expressed by Equation (89). The eigenvalues are found by seeking closed form solutions to Equation (90), and are obtained as Equation (92). The expressions for the buckling loads for any buckling mode are then obtained from the eigenvalues as Equation (102) after some algebraic simplifications. Critical buckling loads are associated with the first buckling modes for which $m = n = 1$. The expression for the critical buckling load coefficient is obtained as Equation (105). The critical buckling load

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coefficient is calculated for varying values of the plate aspect ratio and presented in Table 2, together with previous results obtained by Iyengar [4] and Nwoji et al [23]. Table 2 illustrates the agreement of the present results with previous results obtained by Iyengar [4] and Nwoji et al [23] where Nwoji et al [23] used the double finite Fourier sine integral transform method.

VI. CONCLUSION

(i) The Galerkin-Vlasov method has been proved to be effective for solving the elastic buckling problem of SSCF and SSSS rectangular thin plates subjected to uniform compressive load along the simply supported edges $x = 0$, and $x = a$.

(ii) The displacement function chosen using the Vlasov procedure as the eigenfunctions of a vibrating Euler-Bernoulli beam of equivalent span and support conditions in the x-coordinate direction ensured the satisfaction of all Dirichlet boundary conditions along the simply supported edges ($x = 0$, and $x = a$).

(iii) The Galerkin-Vlasov methodology converts the BVP of elastic stability to an integral equation using the Galerkin-Vlasov variational integral formulation of the problem.

(iv) The Galerkin-Vlasov variational statement (equation), simplifies to a system of homogeneous ordinary differential equations (ODEs) in the unknown function $G_n(y)$.

(v) The system of ODEs is solved by assuming an exponential function as trial function, and this leads to obtaining the basis for linearly independent solutions and the general solution.

(vi) The application of boundary conditions along the edges $y = 0$, and $y = b$ results in a system of homogeneous equations in terms of the integration constants.

(vii) The condition for nontrivial solutions is used on the homogeneous equation to find the elastic stability equation in determinant form.

(viii) The characteristic buckling equation is obtained from expansion and simplification of the determinant as a transcendental equation.

(ix) The eigenvalues of the transcendental equation are obtained using the computational software tools based on iterations, since closed form solutions of some transcendental equations are difficult to obtain, and in most cases have not yet been found.

(x) The expressions found for the elastic stability equation for the considered SSCF and SSSS rectangular thin plates are exact, and identical with previous exact expressions in the literature, thus validating the study results.

(xi) The critical elastic buckling loads found for the considered SSCF and SSSS rectangular thin plates are also exact, since the governing equations were solved for all points in the plate domain, and all boundary conditions are also satisfied by the solution.

(xii) The exact buckling loads obtained are in agreement with results from the literature.

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