Characteristics of Object-oriented Soft Concepts in a Soft Context

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Abstract
We introduced a new type of soft concept called object oriented soft concept (simply, m-concept) based on soft sets, which is independent of the notion of soft concepts in a soft context. The purpose of this work is to study the topological structure in the collection of all the object oriented soft concepts in a soft context. We show that the collection of all the object oriented soft concepts in a soft context is a supratopology. Moreover, we introduce the notions of independent m-concept (object oriented soft concept) and dependent m-concept in a soft context. Using the notions, we show that the set of all independent m-concepts completely determines every m-concept in a given soft context.

Key words and phrases: Formal concepts, soft concepts, object oriented soft concepts, independent m-concepts

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1. INTRODUCTION
FCA (formal concept analysis) was introduced by Wille [11] in 1982, which is an important theory for the research of information structures induced by a binary relation between the set of attributes and objects attributes. The three basic notions of FCA are formal context, formal concept, and concept lattice. A formal context is a kind of information system, which is a tabular form of an object-attribute value relationship [2, 3, 10]. A formal concept is a pair of a set of objects as called the extent and a set of attributes as called the intent.

The concept of soft set was introduced by Molodtsov in 1999 [9], to deal complicated problems and uncertainties. The operations for the soft set theory was introduced by Maji et al. in [4]. Ali et al. [1] proposed new operations modified some concepts introduced by Maji. We have formed a soft context by combining the concepts of the formal context and the soft set defined by the set-valued mapping in [7]. And we introduced and studied the new concepts named soft concepts and soft concepts lattices.

Yao [12] introduced a new concept called an object oriented formal concept in a formal context by using the notion of approximation operations.

We recall that: Let \( (U, A, I) \) be a formal context in formal concept analysis, where \( U \) is a finite nonempty set of objects, \( A \) is a finite nonempty set of attributes and \( I \) is a binary relation between \( U \) and \( A \). For \( x \in U \) and \( y \in A \), if \((x, y) \in I\), also written as \( xIy \). We will denote \( xI = \{ y \in A | xIy \} \); and \( Iy = \{ x \in U | xIy \} \).

And, let us consider two set-theoretic operators,
\[
\Box : P(U) \rightarrow P(A): X^\Box = \{ y \in A | \forall x \in U (xIy \Rightarrow x \in X) \};
\]
\[
\Diamond : P(A) \rightarrow P(U): Y^\Diamond = \{ x \in U | \exists y \in A (xIy \wedge y \in Y) \}.
\]
Then a pair \((X, Y), X \subseteq U, Y \subseteq A\), is called an object oriented formal concept if \( X = Y^\Diamond \) and \( Y = X^\Box \).

Using the facts, we introduced the new notions of object-oriented soft concepts (simply, m-concepts) and studied the notion of m-concepts and basic properties in [8]. The purpose of this work is to study the topological structure in the family of all object-oriented soft concepts. Furthermore, we introduce the notions of independent m-concept and dependent m-concept in a soft context. In particular, we show that the set of all independent m-concepts completely determines every m-concept in a soft context.

2. PRELIMINARIES
A formal context is a triplet \((U, A, I)\), where \( U \) is a non-empty finite set of objects, \( A \) is a nonempty finite set of attributes, and \( I \) is a relation between \( U \) and \( A \). Let \((U, A, I)\) be a formal context. For a pair of elements \( x \in U \) and \( y \in A \), if \((x, y) \in I\), then it means that object \( x \) has attribute \( y \) and we write \( xIy \). The set of all attributes with a given object \( x \in U \) and the set of all objects with a given attribute \( y \in A \) are denoted as the following [10,11]:

\[
x^* = \{ y \in A | xIy \}; \quad y^* = \{ x \in U | xIy \}.
\]

And, the operations for the subsets \( X \subseteq U \) and \( Y \subseteq A \) are defined as:

\[
X^* = \{ y \in A | \forall x \in X, xIy \}; \quad Y^* = \{ x \in U | \forall y \in Y, xIy \}.
\]

In a formal context \((U, A, I)\), a pair \((X, Y)\) of two sets
Let $U$ be a universe set and $A$ be a collection of properties of objects in $U$. We will call $A$ the set of parameters with respect to $U$.

A pair $(F, A)$ is called a soft set [9] over $U$ if $F$ is a set-valued mapping of $A$ into the set $P(U)$ of all subsets of the set $U$, i.e.,

$$ F : A \rightarrow P(U). $$

In other words, for $a \in A$, every set $F(a)$ may be considered as the set of $a$-elements of the soft set $(F, A)$.

Let $U = \{z_1, z_2, \ldots, z_m\}$ be a non-empty finite set of objects, $A = \{a_1, a_2, \ldots, a_n\}$ a non-empty finite set of attributes, and $F : A \rightarrow P(U)$ a soft set. Then the triple $(U, A, F)$ is called a soft context [7].

And, in a soft context $(U, A, F)$, we introduced the following mappings: For each $Z \in P(U)$ and $Y \in P(A)$,

1. $F^+ : P(A) \rightarrow P(U)$ is a mapping defined as $F^+(Y) = \cap_{a \in Y} F(a)$;
2. $F^- : P(U) \rightarrow P(A)$ is a mapping defined as $F^-(Z) = \{a \in A : Z \subseteq F(a)\}$;
3. $Y : P(U) \rightarrow P(U)$ is an operation defined as $\Psi(Z) = F^+ F^-(Z)$.

Then $Z$ is called a soft concept [7] in $(U, A, F)$ if $\Psi(Z) = F^+ F^-(Z) = Z$. The set of all soft concepts is denoted by $sC(U, A, F)$.

In [8], the following operators $F$ and $\overline{F}$ were introduced as follows:

Let $(U, A, F)$ be a soft context. Then for $C \in P(A)$, $X \in P(U)$,

an operator $F : P(A) \rightarrow P(U)$ is defined by $F(C) = \cup_{c \in C} F(c)$;

an operator $\overline{F} : P(U) \rightarrow P(A)$ is defined by $\overline{F}(X) = \{c \in C : F(c) \subseteq X\}$.

Simply, we denote: For $c \in A$ and $x \in U \overline{F}(c) = \overline{F}(c)$ and $\overline{\overline{F}}(x) = \overline{F}(x)$. Obviously, $F(C) = F(c)$ for $c \in A$.

**Theorem 2.1 ([8])** Let $(U, A, F)$ be a soft context, $S, T \subseteq U$ and $B, C \subseteq A$. Then we have:

1. If $S \subseteq T$, then $\overline{F}(S) \subseteq \overline{F}(T)$; if $B \subseteq C$, then $F(B) \subseteq F(C)$;
2. $F \overline{F}(S) \subseteq S$; $\overline{F} F(B) \subseteq B$;
3. $\overline{F} (S \cap T) = \overline{F}(S) \cap \overline{F}(T)$, $F(B \cup C) = F(B) \cup F(C)$;
4. $\overline{F} (S) = \overline{F} F \overline{F}(S)$, $F(B) = F \overline{F} F(B)$.

Let us consider an operator defined as follows: For each $X \in P(U)$ in a soft context $(U, A, F)$,

$\overline{F} : P(U) \rightarrow P(U)$ is an operator defined by $\overline{F}(X) = F \overline{F}(X)$.

Then $X$ is called an object oriented soft concept (simply, m-concept) [8] in $(U, A, F)$ if $\overline{F}(X) = F \overline{F}(X) = X$. The set of all m-concepts is denoted by $m(U, A, F)$.

**Theorem 2.2 ([8])** Let $(U, A, F)$ be a soft context. Then we have:

1. $\overline{F}(X) \subseteq X$ for $X \subseteq U$.
2. If $X \subseteq Y$, then $\overline{F}(X) \subseteq \overline{F}(Y)$.
3. $\overline{F}(\overline{F}(X)) = \overline{F}(X)$ for $X \subseteq U$.
4. $\overline{F}(\emptyset) = \emptyset$.
5. $\overline{F}(X)$ is an m-concept.
6. For $B \subseteq A$, $F(B)$ is an m-concept.
7. For $a \in A$, $F(a)$ is an m-concept.
8. $X$ is an m-concept if and only if there is some $B \subseteq A$ such that $X = F(B)$.

**3. MAIN RESULTS**

We assume that a soft set $(F, A)$ is pure [5], that is, $\cup_{a \in A} F(a) = U$, $\cap_{a \in A} F(a) = \emptyset$, $F(a) \neq \emptyset$ and $F(a) \neq U$ for each $a \in A$.

**Theorem 3.1** Let $(U, A, F)$ be a soft context. Then for $X, Y \in m(U, A, F)$, $\overline{F}(X \cup Y) = \overline{F}(X) \cup \overline{F}(Y)$.

**Proof 3.2** Let $X, Y \in m(U, A, F)$. Then by (8) of Theorem 2.2, there are $B, C \subseteq A$ satisfying $F(B) = X$ and $F(C) = Y$. Then $X \cup Y = F(B) \cup F(C) = F(B \cup C)$, and so again by Theorem 2.2, $X \cup Y$ is also an m-concept. Consequently, $\overline{F}(X \cup Y) = X \cup Y = \overline{F}(X) \cup \overline{F}(Y)$.

**Example 3.3** Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{a, b, c, d, e, f\}$. Consider a soft context $(U, A, F)$ where a set-valued mapping $F : A \rightarrow P(U)$ is defined by

$F(a) = \{1, 2, 4\}; \quad F(b) = \{2, 4, 5\}; \quad F(c) = \{2, 4\}; \quad F(e) = F(f) = \{1, 3, 5\}$.

For $X = \{1, 2, 4\}$ and $Y = \{1, 3, 5\}$, $\overline{F}(X \cap Y) = \overline{F}(\emptyset) = \emptyset$, $\overline{F}(X) \cap \overline{F}(Y) = \{1, 2, 4\} \cap \{1, 3, 5\} = \emptyset$. So, $\overline{F}(X \cap Y) \neq \overline{F}(X) \cap \overline{F}(Y)$.

From Example 3.2, we know that the family $m(U, A, F)$ is not always a topology on $U$.

A family $\sigma$ of $X$ is called a supra topology [6] on $X$ if $\sigma$ satisfies the conditions: (1) $X, \emptyset \in \sigma$; (2) the union of any number of sets in $\sigma$ belongs to $\sigma$.

**Theorem 3.4 ([8])** Let $(U, A, F)$ be a soft context and $\text{Im}(F) = \{F(C) \mid F : P(A) \rightarrow P(U), \ C \in P(A)\}$. Then

1. $\text{Im}(F) = m(U, A, F)$;
2. For $C_1, \ldots, C_n \subseteq A$, $F(C_1) \cup F(C_2) \cup \cdots \cup F(C_n) \in \text{Im}(F)$. 

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Theorem 3.5 Let \((U, A, F)\) be a soft context. Then the family \(m(U, A, F)\) is a supra topology on \(U\).

Proof 3.6 From Theorem 2.2, it is obtained \(U, \emptyset \in m(U, A, F)\). For \(X_1, \ldots, X_n \in m(U, A, F)\), there are \(C_1, \ldots, C_n \subseteq A\) such that \(X_i = F(C_i)\). So, \(X_1 \cup \cdots \cup X_n = F(C_1) \cup \cdots \cup F(C_n) \in m(U, A, F)\). Consequently, \(m(U, A, F)\) is a supra topology on \(U\).

Let \((X, \sigma)\) be a supratopological space and \(B\) a family of subsets in \(X\). For each supraopen set \(G \in \sigma\), \(G\) is a union of any subset of \(B\). Then we will call \(B\) a base for \(\sigma\) [6].

Theorem 3.7 For a soft context \((U, A, F)\), the family \(F_A = \{F(a) | a \in A\}\) is a base for \(m(U, A, F)\).

Proof 3.8 Since the soft set \((F, A)\) is pure, \(\cup_{a \in A} F(a) = U\). Let \(B = \emptyset \subseteq F_A\). Then \(\cup_{F(a) \in B} F(a) = \emptyset\).

For any \(X \in m(U, A, F)\), from (8) of Theorem 2.2, there is some \(B \subseteq A\) such that \(X = \mathbb{F}(B) = \cup_{F(b) \in B} F(b)\). So, the family \(F_A = \{F(a) | a \in A\}\) is a base for \(m(U, A, F)\).

Now, to study the property of \(F_A = \{F(a) | a \in A\}\), we introduce the following concepts:

Definition 3.9 Let \((U, A, F)\) be a soft context. Then for \(Z \in m(U, A, F)\),

1. \(Z\) is said to be dependent on \(m(U, A, F)\) if there exist \(Z_1, \ldots, Z_n \in m(U, A, F)\) satisfying \(Z_i \subseteq Z\) and \(Z = \bigcup Z_i\), \(i = 1, \ldots, n\).
2. \(Z\) is said to be independent of \(m(U, A, F)\) if \(Z\) is not dependent.

We will denote:

\(mD = \{Z \in m(U, A, F) | Z\) is dependent on \(m(U, A, F)\}\);

\(mI = \{Z \in m(U, A, F) | Z\) is independent of \(m(U, A, F)\}\).

Example 3.10 Let \(U = \{1, 2, 3, 4, 5\}\) and \(A = \{a, b, c, d, e\}\). Consider a soft context \((U, A, F)\), where the set-valued mapping \(F : A \rightarrow P(U)\) is defined as follows:

\[F(a) = \{1, 2, 4\}, \quad F(b) = \{1, 2, 4, 5\}, \quad F(c) = \{2, 4\}, \quad F(d) = \{1, 3\}, \quad F(e) = \{1, 5\}.\]

Then,

\(m(U, A, F) = \{\emptyset, \{1, 3\}, \{1, 5\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, U\}\).

For \(X = \{1, 2, 4, 5\} \in m(U, A, F)\), we can take two \(m\)-concepts \(Y = F(c) = \{2, 4\}\) and \(Z = F(e) = \{1, 5\}\) in \(m(U, A, F)\) satisfying \(X \subseteq Y, Z\) and \(X = Y \cup Z\). Hence, \(X\) is dependent, while the \(m\)-concepts \(Y, Z\) are independent.

Theorem 3.11 Let \((U, A, F)\) be a soft context. Then

1. \(\emptyset\) and \(U\) are dependent.
2. \(mD \cap mI = \emptyset\); \(mD \cup mI = m(U, A, F)\).
3. For \(Z \in mD\), there is \(C \subseteq A\) satisfying for \(c \in C\), \(F(c) \subseteq X\) and \(\mathbb{F}(C) = Z\).
4. For \(Z \in mI\), there is \(c \in A\) satisfying \(F(c) = Z\).

Proof 3.12 (1) For the empty set \(\emptyset\), there is \(B = \{Z \in m(U, A, F) | Z \subseteq \emptyset\}\). So, \(\cup_{Z \in B} Z = \emptyset\).

Now, let \(B = \{Z_i \in m(U, A, F) | Z_i \subseteq U, i = 1, \ldots, n\}\). Then \(B = m(U, A, F) - \{U\}\). Since the soft set \((F, A)\) is pure, for \(a \in A\), \(F(a) \in B = m(U, A, F) - \{U\}\) and \(\cup_{a \in A} F(a) = U\) so, \(U\) is dependent.

(2) It is obvious.

(3) For \(Z \in mD\), there are \(Z_1, \ldots, Z_n \in m(U, A, F)\) such that \(Z_i \subseteq Z\) and \(Z = \bigcup Z_i\), \(i = 1, \ldots, n\). From Theorem 2.2, it follows that there are \(C_1, \ldots, C_n \subseteq A\) such that \(F(C_i) = Z_i\). Therefore, \(F(C_1) \subseteq Z\) and \(Z = \bigcup F(C_i) = F(C_i), i = 1, \ldots, n\). Put \(C = \bigcup_i C_i\). Then \(C \subseteq A\) and \(F(C) = Z \supseteq F(c)\) for \(c \in C\).

(4) Let \(Z \in mI\). Then there is \(C \subseteq A\) such that \(F(C) = Z\). Suppose that for every \(c \in C\), \(Z \supseteq F(c)\), which contradicts to \(Z \in mI\). So, there is an element \(d \in C\) satisfying \(Z = F(d)\).

Theorem 3.13 Let \((U, A, F)\) be a soft context. Then for each \(X \in mD\), there is a family \(B \subseteq mI\) satisfying \(X = \bigcup B\).

Proof 3.14 Let an \(m\)-concept \(X\) be dependent. Suppose \(X\) cannot be represented as a union of only elements of \(mI\).

Put \(S = \{X \in mD | X\) cannot be represented as a union of elements of \(mI\}\).

Then, by hypothesis, \(S \neq \emptyset\) and assume that \(|S| = m < |mD|\) where \(|mD|\) is the cardinal number of the set \(mD\).

First, pick up one element \(X\) in \(S\) (say, \(X_1\)). Then since \(X_1 \in mD\), there is a family \(Y_1 = \{Y_{11}, \ldots, Y_{1k}\}\) satisfying \(Y_{1i} \in m(U, A, F), Y_{1i} \subseteq X_1\) and \(X_1 = \bigcup Y_{1i}, i = 1, \ldots, k\).

Additionally, since \(X_1 \in S\), \(Y_1 \cap S \neq \emptyset\). Without the loss of generality, we can choose one dependent \(m\)-concept in \(Y_1 \cap S\), say \(X_2\). Then \(X_1 \supseteq X_2\) and since \(X_2 \in mD\), there is a family \(Y_2 = \{Y_{21}, \ldots, Y_{2m}\}\) such that \(X_2 \supseteq Y_{2k} \in m(U, A, F)\) and \(X_2 = \bigcup Y_{2i}, i = 1, \ldots, m\). And since \(X_2 \in S\), \(Y_2 \cap S \neq \emptyset\).

By repeating this process, finally we can pick up the last element \(X_m\) in \(S\) that satisfies \(X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{m-1} \supseteq X_m\).

Since \(X_m \in mD\), there is a family \(Y_m = \{Y_{mk} | Y_{mk} \in m(U, A, F), 1 = 1, \ldots, r\}\) satisfying \(X_m \supseteq Y_{mk}\) and \(X_m = \bigcup Y_m\).

But, since \(X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m\) and \(|S| = m, S \cap Y_m = \emptyset\).

So, \(X_m\) is not in \(S\).

Since \(X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{m-1} \supseteq X_m\) and \(X_m\) is not in \(S\), \(X_{m-1}\) is also not in \(S\).

For the same reason as \(X_{m-1} \supseteq X_{m-2}\) is also not in \(S\). In the end, it leads to \(S = \emptyset\), which is a contradiction. So, every
dependent $m$-concept can be represented as a union of only independent $m$-concepts of $mI$.

**Theorem 3.15** In a soft context $(U, A, F)$, $mI$ is the smallest base for $m(U, A, F)$.

**Proof 3.16** Let $B$ be a base and $B \subseteq mI$. Then for $X \in mI - B$, there are $S_1, \ldots, S_n \in B$ such that $X = \cup S_i$, which contradicts to $X \not\in mI$. So, $mI$ is the smallest base.

**Theorem 3.17** Let $(U, A, F)$ be a soft context. For $B \subseteq A$, if a set-valued mapping $\varphi : B \rightarrow mI$ defined by $\varphi(b) = F(b)$ for $b \in B$ is surjective, then $\varphi(B) = \{F(b) \mid b \in B\}$ is a base for $m(U, A, F)$.

**Proof 3.18** Obvious.

**Remark 3.19** Let $(U, A, F)$ be a soft context.

For $mI$,

$m(U, A, F) = \{\cup M \mid M \subseteq mI\}$.

For $F_A = \{F(a) \mid a \in A\}$,

$m(U, A, F) = \{\cup M \mid M \subseteq F_A\}$. 

For $B \subseteq A$ and a surjective mapping $\varphi : B \rightarrow mI$ defined by $\varphi(b) = F(b)$ for $b \in B$,

$m(U, A, F) = \{\cup M \mid M \subseteq \varphi(B)\}$. 

For $C \subseteq A$ and a bijective mapping $\psi : C \rightarrow mI$ defined by $\psi(c) = F(c)$ for $c \in B$,

$m(U, A, F) = \{\cup M \mid M \subseteq \psi(C)\}$.

In summary, we have the size relationships for the above bases as follows: For $B, C \subseteq A$,

$|mI| = |\psi C| \leq |\varphi B| \leq |F_A| \leq |m(U, A, F)|$

**4. CONCLUSION**

We studied the notion of $m$-dependent and $m$-independent soft concepts in a given soft context. Additionally, we showed that every $m$-dependent soft concept is generated by some $m$-independent soft concepts. In the next study, we will study the various characteristics of such notions and apply these results to object oriented concepts of a formal context.

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