Related Fixed Point Theorems on Two Complete and Compact G-Metric Spaces

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Abstract

New results concerning the related fixed point theorems on two complete G-metric spaces are proved and deduced some corollaries. We prove also a related fixed point theorems on two compact G-metric spaces.

Keyword: Fixed point, Complete G-metric spaces, Compact G-metric spaces

1. INTRODUCTION

In [7],[8], Fisher proved some related fixed point theorems in two complete metric spaces which is as follows:

**Theorem 1.1.** Let \((X, d)\) and \((Y, \rho)\) be a complete metric spaces. If \(T\) is a mapping from \(X\) into \(Y\) and \(S\) is a mapping from \(Y\) into \(X\) satisfying the following conditions:

\[
\begin{align*}
\rho(Tx, Tsy) & \leq c \max\{d(x, Sy), \rho(y, Tx), \rho(y, STy)\} \\
\rho(Sy, STx) & \leq c \max\{\rho(y, Tx), d(Sy, St), d(x, STx)\}
\end{align*}
\]

for all \(x, y \in X\) where \(c \in [0, 1]\), then \(ST\) have a unique fixed point \(w\) in \(Y\). Further \(Tz = w\) and \(Sw = z\).

In [30], Popa extended the results of Fisher. Besides, Cho [6], extended and improved the results of Fisher [7],[8], and Popa [30]. Recently, related fixed point theorems on three complete metric spaces have been studied by Fisher and Rao [28-30], Nung [24], Jain and Rao[10-12], Jain and Dixit[9].

In 2006 Mustafa, and Sims, introduced the notion of generalized metric space called G-metric space [15]. In this generalization to every triplet of elements in the space assigned a non-negative real number. An analysis of the structure of these spaces was done in details in [15]. Subsequently, several authors proved many kind of fixed point theorems for contractive type mapping and expansive mapping in generalized metric spaces (see [1]-[3],[4-5],[13-14],[16-23],[25],[27],[31]). On the other hand, Rao [31], obtained the related fixed point theorems on three complete G-metric spaces.

In the first part of this paper, we prove some results concerning the related fixed point theorems on two complete G-metric spaces and deduce some corollaries. In the second part, we prove also a related fixed point theorems on two compact G-metric spaces. The results of this paper are new in G-metric spaces.

2. PRELIMINARIES

We recall some basic definitions and results which are important in the sequel. We refer to [19], for details on the following notions. Throughout this paper, \(\mathbb{R}\) denotes the set of all real numbers, \(\mathbb{R}^+\) denotes the set of nonnegative reals and \(\mathbb{N}\) denotes the set of natural numbers.

**Definition 2.1.** Let \(X\) be a non empty set and \(G : X \times X \times X \rightarrow \mathbb{R}^+\) be a function satisfying the following axioms:

\[
\begin{align*}
(G1) & \quad G(x, y, z) = 0 \text{ if } x = y = z, \\
(G2) & \quad 0 < G(x, x, y), \text{ for all } x, y \in X, \text{ with } x \neq y, \\
(G3) & \quad G(x, x, y) \leq G(x, y, z), \text{ for all } x, y, z \in X, \text{ with } z \neq y, \\
(G4) & \quad G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \text{ (symmetry in all three variables),} \\
(G5) & \quad G(x, y, z) \leq G(x, a, a) + G(a, y, z), \text{ for all } x, y, z, a \in X \text{ (rectangle inequality).}
\end{align*}
\]

Then the function \(G\) is called a generalized metric, or more specifically a G-metric on \(X\), and the pair \((X, G)\) is called a G-metric space.

**Example 2.1.** Define \(G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+\) by \(G(x, y, z) = |x - y| + |y - z| + |z - x|\), for all \(x, y, z \in X\). Then it is clear that \((\mathbb{R}, G)\) is a G-metric space.

**Proposition 2.1.** Let \((X, G)\) be a G-metric space. Then for any \(x, y, z \text{ and } a \in X\), it follows that:

\[
\begin{align*}
(1) & \quad G(x, y, z) = 0 \text{ then } x = y = z, \\
(2) & \quad G(x, y, z) \leq G(x, x, y) + G(x, x, z), \\
(3) & \quad G(x, y, z) \leq 2G(y, x, x).
\end{align*}
\]

**Definition 2.2.** Let \((X, G)\) be a G-metric space, and \((x_n)\) be a sequence of points of \(X\), we say that \((x_n)\) is G-convergent
to \( x \in X \) if for any \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that
\[ G(x,x_n,x_m) < \epsilon, \] for all \( n,m \geq N \).

**Proposition 2.2.** Let \((X,G)\) be a G-metric space. Then the following are equivalent:

1. \((x_n)\) is G-convergent to \( x \).
2. \(G(x_n,x_n,x) \to 0\), as \( n \to \infty \).
3. \(G(x_n,x,x) \to 0\), as \( n \to \infty \).

**Definition 2.3.** Let \((X,G)\) be a G-metric space, a sequence \((x_n)\) is called G-Cauchy if given \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( G(x_n,x_m,x_l) < \epsilon \), for all \( n,m,l \geq N \).

**Definition 2.4.** A G-metric space \((X,G)\) is said to be G-complete if every G-Cauchy sequence in \((X,G)\) is G-convergent in \((X,G)\).

**Definition 2.5.** A G-metric space \((X,G)\) is said to be a compact G-metric space if it is G-complete and G-totally bounded.

**Definition 2.6.** Let \((X,G_1)\) and \((Y,G_2)\) be complete G-metric spaces, and let \( f : (X,G_1) \to (Y,G_2) \) be a function, then \( f \) is said to be G-continuous at a point \( a \in X \), if given \( \epsilon > 0 \), there exists \( \delta > 0 \), such that \( x_1, x_2 \in X, G_1(a,x_1,x_2) < \delta \) implies \( G_2(f(a),f(x_1),f(x_2)) < \epsilon \).

A function \( f \) is G-continuous on \( X \) if and only if, it is G-continuous at all \( a \in X \).

**Proposition 2.3.** Let \((X,G)\) be a G-metric space. Then the function \( G(x,y,z) \) is continuous in all variables.

### 3. RELATED FIXED POINT THEOREMS ON COMPLETE G-METRIC SPACES

Our main result follows:

Let \( \mathbb{S} \) be the set of all continuous real functions \( g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that:

(i) \( g(0,0,0) = 0 \)

(ii) If \( u^2 \leq g(uv,0,0) \) or \( u^2 \leq g(0,uw,0) \) or \( u^2 \leq g(0,0,uv) \), for all \( u,v \in \mathbb{R}^+ \), then there exists \( 0 \leq c < 1 \) such that \( u \leq \frac{1}{4}cv \).

**Example 3.1.** If we define a function \( g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) by the following:

(a) \( g(u,v,w) = \frac{1}{4}c\max\{uw,vu,uv\} \), for all \( u,v,w \in \mathbb{R}^+ \), where \( 0 \leq c < 1 \).

(b) \( g(u,v,w) = \frac{1}{4}(auw+buv+cuv) \), for all \( u,v,w,a,b,c \in \mathbb{R}^+ \).

Then \( g \in \mathbb{S} \).

**Theorem 3.1.** Let \((X,G_1)\) and \((Y,G_2)\) be complete G-metric spaces, and \( T \) be a mapping of \( X \) into \( Y \) and let \( S \) be a mapping of \( Y \) into \( X \) satisfying the inequalities:

\[
G^2_2(Tx,TSy_1,TSy_2) \leq g(G_2(y_1,TSy_1,TSy_2)G_2(y_1,y_2,Tx),
G_2(y_1,y_2,Tx)G_1(x,STy_1,STy_2),
G_1(x,STy_1,STy_2)G_2(y_1,TSy_1,TSy_2))
\]

(3.1)

\[
G^2_1(STy_1,STy_2) \leq g(G_1(x,x,STx)G_1(x,STy_1,STy_2),
G_1(x,STy_1,STy_2)G_2(y_1,y_2,Tx),
G_2(y_1,y_2,Tx)G_1(x,STx))
\]

(3.2)

for all \( x \in X \) and \( y_1, y_2 \in Y \), where \( g \in \mathbb{S} \). Then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further, \( Tz = w \) and \( Sw = z \).

**Proof.** We define the sequences \((x_n)\) in \( X \) and \((y_n)\) in \( Y \) by
\[
x_n = (ST)^n x,\ y_n = T(S)^{n-1} x, \quad \text{for } n = 1,2,... \quad \text{We will assume that } x_n \neq x_{n+1} \text{ and } y_n \neq y_{n+1} \text{ for all } n. \]

Applying the inequality (3.1) and using property (ii), we have

\[
G_2(y_n,y_{n+1}, y_{n+1}) \leq \frac{1}{4}G_1(x_{n-1},x_n,x_n)G_2(y_n,y_{n+1}, y_{n+1}) \]

(3.3)

Similarly, applying the inequality (3.2),

\[
G_1(x_n,x_n+1) = G_1(Sy_n, Sy_n, STx_n) \leq g(G_1(x_n,x_{n+1})G_1(x_n, Sy_n, Sy_n),
G_1(x_n, Sy_n, Sy_n)G_2(y_n,Tx_n),
G_2(y_n,y_{n+1}, Tx_n)G_1(x_n, x_{n+1}))) \leq g(G_1(x_n,x_{n+1})G_1(x_n,x_n),
G_1(x_n,x_{n+1})G_2(y_n,y_{n+1}),
G_2(y_n, y_{n+1})G_1(x_n, x_{n+1}))
\]

Using property (ii) and the Proposition(2.2), we have

\[
G_2^2(x_n,x_{n+1}) \leq \frac{1}{4}G_2^2(y_n,y_{n+1})G_1(x_n,x_{n+1}) \leq \frac{1}{4}G_1(x_n,x_{n+1}) \leq \frac{1}{2} \]

\[
G_2^2(y_n,y_{n+1}) \leq \frac{1}{4}G_2^2(y_n,y_{n+1}) \leq \frac{1}{2}
\]

\[
cG_2(y_n,y_{n+1}, y_{n+1})
\]
\[ G_1(x_n, x_{n+1}, x_{n+1}) \leq cG_2(y_n, y_{n+1}, y_{n+1}) \]  
(3.4)

Now it follows from the inequalities (3.3) and (3.4) that
\[ G_1(x_n, x_{n+1}, x_{n+1}) \leq \frac{1}{4}c^2G_1(x_{n-1}, x_n, x_n). \]

Hence, by induction we get
\[ G_1(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{1}{4}\right)^n c^{2n}G_1(x, x_1, x_1), \text{ for } n = 1, 2, \cdots \]
(3.5)

So \((x_n)\) and \((y_n)\) are G-Cauchy sequences with limits \(z\) in \(X\) and \(w\) in \(Y\). Using the inequality (3.1), we have

\[ G_2^2(Tz, y_n, y_n) = G_2^2(Tz, TSy_{n-1}, TSy_{n-1}) \leq g(G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1})G_2(y_{n-1}, y_{n-1}, Tz), \]
\[ G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, Sy_{n-1}, Sy_{n-1}, G_1(z, Sy_{n-1}, Sy_{n-1}, Sy_{n-1})), \]
\[ G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, x_{n-1}, x_{n-1}), \]
\[ G_1(z, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_{n-1}, y_n)). \]

Thus
\[ G_2^2(Tz, w, w) \leq g(0, 0, 0) = 0, \]

it follows that \(G_2(Tz, w, w) = 0\), hence \(w = Tz\). Using the inequality (3.2), we have

\[ G_2^2(Sw, Sw, x_n) = G_2^2(Sw, Sw, STx_{n-1}) \leq g(G_1(x_{n-1}, x_{n-1}, STx_{n-1})G_1(x_{n-1}, Sw, Sw), \]
\[ G_1(x_{n-1}, Sw, Sw)G_2(w, w, Tz_{n-1}), \]
\[ G_2(w, w, Tz_{n-1})G_1(x_{n-1}, x_{n-1}, STx_{n-1}), \]

Letting \(n\) tends to infinity and using (i), we have
\[ G_2^2(Sw, Sw, x_n) \leq g(0, 0, 0) = 0, \]
and it follows that \(z = Sw\). Thus \(STz = Sw = z, TSz = Tz = w\), and so \(ST\) has a fixed point \(z\) and \(TS\) has a fixed point \(w\). To prove uniqueness, suppose that \(ST\) has a second fixed point \(z_1\) and \(TS\) has a second fixed point \(w_1\). Then applying the inequality (3.1) and using property (ii), we have

\[ G_2^2(w, w_1, w_1) = G_2^2(TSw, TSw, TSw) \leq \frac{1}{4}c^2G_1(Sw, Sw, Sw, Sw) \]
\[ G_2^2(w, w_1, w_1) \leq \frac{1}{4}c^2G_1(Sw, Sw, Sw, Sw) \]

Further, applying the inequality (3.2) and using property (ii), we have

\[ G_2^2(Sw, Sw, Sw) = G_2^2(STSw, STSw, STSw) \]
\[ g(G_1(Sw_1, Sw_1, Sw_1), G_1(Sw_1, Sw_1, Sw_1)), \]
\[ G_1(Sw_1, Sw_1, Sw_1)G_2(TSw, TSw, TSw), \]
\[ G_2(TSw, TSw, TSw)G_1(Sw_1, Sw_1, Sw_1)) \]
\[ \leq g(0, G_1(Sw, Sw, Sw)G_2(w, w, w)), \]

which implies that

\[ G_2^2(Sw, Sw, Sw) \leq \frac{1}{4}c^2G_2(w, w, w)G_1(Sw, Sw, Sw), \]
\[ G_1(Sw, Sw, Sw), \]
\[ G_2(Sw, Sw, Sw) \leq \frac{1}{4}c^2G_2(w, w, w), \]

again by using the Proposition (2.2), we get,
\[ \frac{1}{2}G_1(Sw, Sw, Sw) \]
\[ \leq G_1(Sw, Sw, Sw) \leq \frac{1}{4}c^2G_2(w, w, w) \leq \frac{1}{2}cG_2(w, w_1), \]
\[ G_1(Sw, Sw, Sw, Sw) \leq cG_2(w, w, w_1). \]
(3.8)

Now it follows from the inequalities (3.6) and (3.8) that

\[ G_2(w, w_1, w_1) \leq \frac{1}{4}cG_1(Sw, Sw, Sw) \]
\[ \leq \frac{1}{4}c^2G_2(w, w, w) \leq cG_2(w, w, w), \]
and so \(w = w_1\) since \(c < 1\). The fixed point \(w\) of \(TS\) must be a unique. Now \(TSz = z_1\) implies \(TSz_1 = Tz_1\) and so \(Tz_1 = w\). Thus \(z = STz = Sw = STz_1 = z_1\), proving that \(z\) is a unique fixed point of \(ST\). Thus the proof of the Theorem is completes.

We have the following Corollaries:

**Corollary 3.2.** Let \((X, G_1)\) and \((Y, G_2)\) be complete G-metric spaces, and \(T, S, B\) be a mapping of \(X\) into \(Y\) and let \(S\) be a mapping of \(X\) into \(X\) satisfying the inequalities:

\[ G_2^2(Tx, SY_{y_1}, TSy_{y_2}) \leq \frac{1}{4}c^2\max\{G_1(x, x, STx), G_1(x, y_1, Sy_{y_2}), G_1(x, Sy_{y_2})G_2(y_1, TSy_{y_1}, TSy_{y_2})\} \]

\[ G_2^2(y_1, y_2, y_1, x_1, x_2), G_1(x, y_1, y_2, Tx)G_1(x, x, Sy_{y_2}), \]
\[ G_1(x, Sy_{y_2})B_2(y_1, TSy_{y_1}, TSy_{y_2}) \]

for all \(x, y_1, y_2, z\) in \(X\) and \(y_1, y_2, z\) in \(Y\). 0 \(\leq c < 1\). Then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(z = w = z\).

**Proof.** It is immediate to see that, if we take a function \(g : R^+ \times R^+ \times R^+ \rightarrow R^+\), by \(g(u, v, w) = \frac{1}{4}c\max\{uw, vu, wv\}\), for
all \( u, v, w \in \mathbb{R}^+ \), where \( 0 \leq c < 1 \), then from Example (3.1)(a) it follows that \( g \in \mathcal{G} \) and by the Theorem (3.1), the Corollary follows.

**Corollary 3.3.** Let \((X, G_1)\) and \((Y, G_2)\) be complete \(G\)-metric spaces, and \(T\) be a mapping of \(X\) into \(Y\) and let \(S\) be a mapping of \(Y\) into \(X\) satisfying the inequalities:

\[
G_2^2(Tx, TSy_1, TSy_2) \leq \frac{1}{4} \left( a_1G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx) + b_1G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2) + c_1G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2) \right)
\]

\[
G_2^2(Sy_1, Sy_2, STx) \leq \frac{1}{4} \left( a_2G_1(x, x, STx)G_1(x, Sy_1, Sy_2) + b_2G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx) + c_2G_2(y_1, y_2, Tx)G_1(x, x, STx) \right)
\]

for all \(x \in X\) and \(y_1, y_2 \in Y\), \(a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+\) with \((a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1\). Then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

Now, we give an example to illustrate Theorem(3.1).

**Example 3.2.** Let \(X = Y = [1, \infty)\), we define on \(X\) and \(Y\) the \(G_1\)-metric space and the \(G_2\)-metric space as follows:

\[
G_1(x_1, x_2, x_3) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}, \text{ with } x_1, x_2, x_3 \in X
\]

\[
G_2(y_1, y_2, y_3) = \frac{\sqrt{2}}{16} \max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}, \text{ with } y_1, y_2, y_3 \in Y
\]

Let \(T\) and \(S\) defined by \(Tx = 2x - 1\) and \(Sy = y\), we have

\[
G_2^2(Tx, TSy_1, TSy_2) = G_2^2(Tx, Ty, Ty) = \left( \frac{\sqrt{2}}{16} \right)^2 |Tx - Ty| |Ty - Ty| = \frac{1}{4} \frac{\sqrt{2}}{2} G_1(x, Sy, Sy)G_2(y, Ty, Ty)
\]

\[
= \frac{1}{4} c \max\{0, 0, G_1(x, Sy, Sy)G_2(y, Ty, Ty)\} = g(0, 0, G_1(x, Sy, Sy)G_2(y, Ty, Ty))
\]

then \(ST\) and \(TS\) have the unique fixed point 1.

**Theorem 3.4.** Let \((X, G_1)\) and \((Y, G_2)\) be complete \(G\)-metric spaces, and \(T\) be a mapping of \(X\) into \(Y\) and let \(S\) be a mapping of \(Y\) into \(X\) satisfying the inequalities:

\[
G_2^3(Tx, TSy_1, TSy_2) \leq \frac{1}{4} c_1 \max\{G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)G_2(y_1, TSy_1, TSy_2)G_2(y_1, TSy_1, TSy_2) \}
\]

\[
G_2^3(Sy_1, Sy_2, STx) \leq \frac{1}{4} c_2 \max\{G_2(y_1, y_2, Tx)G_1(x, x, STx)G_1(x, x, STx), G_1(x, x, STx)G_1(x, x, STx)G_1(x, x, STx) \}
\]

for all \(x \in X\) and \(y_1, y_2 \in Y\), where \(0 \leq c_1, c_2 < 1\). Then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

**Proof.** We define the sequences \((x_n)\) in \(X\), and \((y_n)\) in \(Y\), by \(x_0 = (ST)^n x, y_0 = T(ST)^{n-1} x, n = 1, 2, \ldots\) We will assume that \(x_n \neq x_{n+1}\) and \(y_n \neq y_{n+1}\) for all \(n\). Applying the inequality (3.9), we have

\[
G_2^3(y_n, y_{n+1}, y_{n+1}) = G_2^3(Tx_{n-1}, TSy_n, TSy_n) \leq \frac{1}{4} c_1 \max\{G_1(x_{n-1}, Sy_n, Sy_n)G_2(y_n, TSy_n, TSy_n)G_2(y_n, TSy_n, TSy_n) \}
\]

\[
G_2^3(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4} c_1 \max\{G_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})G_2(y_n, y_{n+1}, y_{n+1}) \}
\]

It follows that

\[
G_2^3(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4} c_1 G_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})G_2(y_n, y_{n+1}, y_{n+1})
\]
Applying the inequality (3.10), and using the Proposition (2.2), we get

\[
G_2(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4} c G_1(x_{n-1}, x_n)
\]

(3.11)

Now it follows from the inequalities (3.11) and (3.12) that

\[
G_1(x_n, x_{n+1}, x_{n+1}) \leq \frac{1}{4} c_2 G_2(y_{n+1}, y_{n+1}, y_{n+1}) \leq \frac{1}{2} c_2 G_2(y_{n+1}, y_{n+1}, y_{n+1}).
\]

(3.12)

Hence, by induction we get

\[
G_1(x_n, x_{n+1}, x_{n+1}) \leq \left(\frac{1}{4}\right)^n (c_2 c_1)^n G_1(x_1, x_1), \text{ for } n = 1, 2, \ldots
\]

Since \(c_2 c_1 < 1\), it follows that \(x_n\) and \(y_n\) are G-Cauchy sequences with limits \(z\) in \(X\) and \(w\) in \(Y\). Using the inequality (3.9), we have

\[
G_2^2(Tz, y_n, y_n) = G_2^2(Tz, TSy_{n-1}, TSy_{n-1})
\]

\[
\leq \frac{1}{4} c_1 \max \{G_1(x_n, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_{n-1}, TSy_{n-1})G_2(y_{n-1}, TSy_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, TSy_{n-1})\}
\]

\[
\leq \frac{1}{4} c_1 \max \{G_1(x_n, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})G_2(y_{n-1}, y_{n-1}, y_{n-1})\}
\]

(3.13)
Applying the inequality (3.10), Proposition (2.2) we have

$$G_1^3(Sw, Sw, Sw_1) = G_1^3(STSw, STSw, STSw_1) \leq \frac{1}{4} c_2 \max \{G_2(TSw, TSw, TSw_1)G_1(Sw, Sw_1, STSw_1)G_1(Sw, Sw, STSw),$$

$$G_1(Sw_1, STSw, STSw)G_2(TSw, TSw, TSw_1)G_1(Sw_1, STSw, STSw),$$

$$G_1(Sw_1, Sw_1, STSw_1)G_1(Sw_1, STSw, STSw_1)\} \leq \frac{1}{4} c_2 \max \{0, G_1(Sw_1, Sw, Sw)G_1(Sw_1, Sw, Sw)G_2(w, w, w_1), 0\}$$

$$G_1^3(Sw, Sw, Sw_1) \leq \frac{1}{4} c_2 G_2(w, w, w_1)G_1(Sw, Sw, Sw_1)G_1(Sw, Sw, Sw_1) \leq \frac{1}{4} c_2 G_2(w, w, w_1) \leq \frac{1}{4} c_2 G_2(w, w, w_1)$$

$$G_1(Sw_1, Sw_1, Sw_1) \leq G_1(Sw, Sw_1, Sw_1) \leq G_1(Sw, Sw_1, Sw_1) \leq c_2 G_2(w, w_1) \tag{3.15}$$

Now it follows from the inequalities (3.14) and (3.16) that

$$G_2(w, w_1, w_1) \leq \frac{1}{4} c_1 G_1(Sw, Sw_1, Sw_1) < \frac{1}{4} c_1 c_2 G_2(w, w_1, w_1) < G_2(w, w_1, w_1)$$

and so $w = w_1$ since $c_1 c_2 < 1$. The fixed point $w$ of $TS$ must be a unique. Now $TSz_1 = z_1$ implies $TSTz_1 = Tz_1$ and so $Tz_1 = w$. Thus $z = STz = Sw = STz_1 = z_1$, proving that $z$ is the unique fixed point of $ST$. This completes the proof of the

**Corollary 3.5.** Let $(X, G_1)$ and $(Y, G_2)$ be complete G- metric spaces, and $T$ be a mapping of $X$ into $Y$ and let $S$ be a mapping of $Y$ into $X$ satisfying the inequalities:

$$G_2^3(Tx, TSy_1, TSy_2) \leq \frac{1}{4} \left( a_1 G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)G_2(y_1, TSy_1, TSy_2) +$$

$$b_1 G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Ty) + c_1 G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx)G_2(y_1, y_2, Ty) \right)$$

$$G_2^3(Sy_1, Sy_2, STx) \leq \frac{1}{4} \left( a_2 G_2(y_1, y_2, Tx)G_1(x, x, STx)G_1(x, x, STx) + b_2 G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx)G_1(x, x, STx) +$$

$$c_2 G_1(x, x, STx)G_1(x, x, STx)G_1(x, x, STx) \right)$$

for all $x \in X$ and $y_1, y_2 \in Y$, $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$. Then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further $Tz = w$ and $Sw = z$.

**Example 3.3.** Let $X = Y = [1, \infty)$, we define on $X$ and $Y$ the $G_1$-metric space and the $G_2$-metric space as follows:

$$G_1(x_1, x_2, x_3) = \max \{ |x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1| \}, \text{ with } x_1, x_2, x_3 \in X$$

$$G_2(y_1, y_2, y_3) = \frac{\sqrt{7}}{4} \max \{ |y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1| \}, \text{ with } y_1, y_2, y_3 \in Y.$$

Let $T$ and $S$ defined by $Tx = 3x - 2$ and $Sy = y$, we have

$$G_2^3(Tx, TSy, TSy) = G_2^3(Tx, Ty, Ty) = 3\left( \frac{\sqrt{7}}{4} \right)^2 |x - y| |Tx - Ty| = \frac{1}{4} \sqrt{7} G_1(x, Sy, Sy)G_2(y, Ty, Ty)G_2(y, Ty, Ty)$$

$$= \frac{1}{4} \max \{ G_1(x, Sy, Sy)G_2(y, Ty, Ty)G_2(y, Ty, Ty), 0, 0 \}$$

then $ST$ and $TS$ have a unique fixed point 1.
4. RELATED FIXED POINT THEOREMS ON COMPACT G-METRIC SPACES

In this section, we prove an analogous results for compact G-metric spaces. Let $\mathcal{G}^*$ denote the set of all real functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(i) If $u^2 < g(uv, 0, 0)$ or $u^2 < g(0, uv, 0)$ or $u^2 < g(0, 0, uv)$ for all $u, v \in \mathbb{R}^+$, then $u < \frac{1}{2}v$.

**Theorem 4.1.** Let $(X, G_1)$ and $(Y, G_2)$ be compact G-metric spaces, and $T$ be a continuous mapping of $X$ into $Y$ and let $S$ be a continuous mapping of $Y$ into $X$ satisfying the inequalities:

\[
G_2^2(Tx, TSy_1, TSy_2) < g(G_2(y_1, TSy_1, TSY_2)G_2(y_1, y_2, Tx), G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2), \quad (4.1)
\]

\[
G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSY_2)) \quad \text{for all } x \in X \text{ and } y_1, y_2 \in Y \text{ with } x \neq Sy_1 \text{ and } x \neq Sy_2, \text{ where } g \in \mathcal{G}^*, \text{ and}
\]

\[
G_2^2(Sy_1, Sy_2, STz) < g(G_1(x, STz)G_1(x, Sy_1, Sy_2), G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx), \quad (4.2)
\]

\[
G_2(y_1, y_2, Tx)G_1(x, x, STx) \quad \text{for all } x \in X \text{ and } y_1, y_2 \in Y, \text{ where } g \in \mathcal{G}^* \text{ with } y_1 \neq Tx, y_2 \neq Tx. \text{ Then ST has a unique fixed point } z \text{ in X and TS has a unique fixed point w in Y. Further, } Tz = w \text{ and } Sw = z.
\]

**Proof.** Let $\psi : X \to \mathbb{R}^+$ defined by $\psi(x) = G_1(x, STx, STx)$ is G-continuous on $X$. Since $X$ is compact, there exists a point $u$ in $X$ such that $\psi(u) = G_1(u, STu, STu) = \min\{G_1(x, STx, STx) \mid x \in X\}$. Now suppose that $Tu \neq TSTu$, then $u \neq STu$. Put $y_1 = y_2 = Tu, x = Sy = STu$ in the inequality (4.2), we have

\[
G_2^2(STu, STu, STSTu) < g(G_1(STu, STu, STSTu)G_1(STu, STu, STu), G_1(STu, STu, STu)G_2(Tu, Tu, TSTu), G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)) \leq g(0, 0, G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu))
\]

Using condition (i) and Proposition(2.2) we have

\[
G_2^2(STu, STu, STSTu) < \frac{1}{2}G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)
\]

\[
G_1(STu, STu, STSTu) < \frac{1}{2}G_2(Tu, Tu, TSTu) < G_2(Tu, TSTu, TSTu)
\]

Put $y_1 = y_2 = Tu, x = u$ in the inequality (4.1), we have

\[
G_2^2(Tu, TSTu, TSTu) < g(G_2(Tu, TSTu, TSTu)G_2(Tu, Tu, Tu), G_2(Tu, Tu, Tu)G_1(u, STu, STu), G_1(u, STu, STu)G_2(Tu, TSTu, TSTu)) \leq g(0, 0, G_1(u, STu, STu)G_2(Tu, TSTu, TSTu))
\]

But using condition (i), we get

\[
G_2^2(Tu, TSTu, TSTu) < \frac{1}{2}G_1(u, STu, STu)G_2(Tu, TSTu, TSTu),
\]

\[
G_2(Tu, TSTu, TSTu) < \frac{1}{2}G_1(u, STu, STu)
\]

\[
\frac{1}{2}G_1(STu, STSTu, STSTu) \leq G_1(STu, STu, STSTu) < \frac{1}{2}G_1(u, STu, STu)
\]

\[
G_1(STu, STSTu, STSTu) < G_1(u, STu, STu).
\]

Hence $\psi(STu) < \psi(u)$, and this gives us a contradiction. So $TSTu = Tu$. If putting $Tu = w$ and $Sw = z$, then we get $ST(STu) = S(TSTu) = STu = Sw = z$, and $w = Tu = TST(Tu) = T(STu) = Tz$. Thus, $Sw = z$ is a fixed point of $ST$ and $Tz = w$ is a fixed point of $TS$. To prove uniqueness, suppose that $ST$ has a second distinct fixed point $z_1$. Then applying the inequality (4.2) and using condition (i), we have

\[
G_1(z, z, z_1) = G_1(STz, STz, STz_1) < g(G_1(z_1, z_1, STz_1)G_1(z_1, z, z),
\]
G_1(z_1, z_1, z_1)G_2(Tz, Tz, Tz_1), G_2(Tz, Tz, Tz_1)G_1(z_1, z_1, STz_1)).

It follows that

\[ G_1^2(z, z, z_1) < \frac{1}{2} G_2(Tz, Tz, Tz_1)G_1(z_1, z_1, STz_1) \]

Further, applying the inequality (4.1) and using condition (i) we have,

\[ G_1(z_1, z_1, z_1) < \frac{1}{2} G_2(Tz, Tz, Tz_1) \]

This completes the proof of the Theorem.

**Corollary 4.2.** Let \((X, G_1)\) and \((Y, G_2)\) be compact G-metric spaces, and \(T\) be a continuous mapping of \(X\) into \(Y\) and let \(S\) be a continuous mapping of \(Y\) into \(X\) satisfying the inequalities:

\[ G_2(Tx, TSy_1, TSy_2) < \frac{1}{2} \max \{G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx), G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2)\} \]

for all \(x \in X\) and \(y_1, y_2 \in Y\) with \(x \neq Sy_1\) and \(x \neq Sy_2\), and

\[ G_1(Sy_1, Sy_2, STx) < \frac{1}{2} \max \{G_1(x, x, STx)G_1(x, Sy_1, Sy_2), G_1(x, x, Sy_2)G_2(y_1, y_2, Tx)\} \]

for all \(x \in X\) and \(y_1, y_2 \in Y\) with \(y_1 \neq Tx, y_2 \neq Tx\). Then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

**Corollary 4.3.** Let \((X, G_1)\) and \((Y, G_2)\) be compact G-metric spaces, and \(T\) be a continuous mapping of \(X\) into \(Y\) and let \(S\) be a continuous mapping of \(Y\) into \(X\) satisfying the inequalities:

\[ G_2(Tx, TSy_1, TSy_2) < \frac{1}{2} (a_1G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx) + b_1G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2) + c_1G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)) \]

for all \(x \in X\) and \(y_1, y_2 \in Y\) with \(x \neq Sy_1\) and \(x \neq Sy_2\), and

\[ G_1(Sy_1, Sy_2, STx) < \frac{1}{2} (a_2G_1(x, x, STx)G_1(x, Sy_1, Sy_2) + b_2G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx) + c_2G_2(y_1, y_2, Tx)G_1(x, x, STx)) \]

for all \(x \in X\) and \(y_1, y_2 \in Y\), with \(y_1 \neq Tx, y_2 \neq Tx\), and \(a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+\) with \((a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1\). Then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

We give an example to support Theorem(4.1).

**Example 4.1.** Let \(X = Y = [0, 1]\), we define on \(X\) and \(Y\) the \(G_1\)-metric space and the \(G_2\)-metric space as follows:

\[ G_1(x_1, x_2, x_3) = \max \{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}, \text{ with } x_1, x_2, x_3 \in X \]

\[ G_2(y_1, y_2, y_3) = \frac{\sqrt{3}}{9} \max \{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}, \text{ with } y_1, y_2, y_3 \in Y \]
Let $T$ and $S$ defined by $Tx = \frac{3}{4}x^2$ and $Sy = y$, we have
\[
G_2^2(Tx,TSy,TSy) = G_2^2(Tx,Ty,Ty) \leq \frac{3}{2} \sqrt{\frac{5}{9}} |x - y| |Tx - Ty| = \frac{1}{2} \sqrt{\frac{5}{3}} G_1(x, Sy, Sy)G_2(y,Ty,Ty)
\]
\[< \frac{1}{2} \max \{0, 0, G_1(x, Sy, Sy)G_2(y,Ty,Ty)\} = g(0, 0, G_1(x, Sy, Sy)G_2(y,Ty,Ty)),\]
then $ST$ and $TS$ have the unique fixed point $0$.

**Theorem 4.4.** Let $(X, G_1)$ and $(Y, G_2)$ be compact $G$-metric spaces, and $T$ be a continuous mapping of $X$ into $Y$ and let $S$ be a continuous mapping of $Y$ into $X$ satisfying the inequalities:
\[
G_2^2(Tx,TSy,TSy) < \frac{1}{2} \max \{G_1(x, Sy, Sy)G_2(y,TSy,TSy)G_2(y1,TSy1,TSy2),
\]
\[
G_2(y1,y2,Tx)G_1(x, Sy, Sy)G_2(y1,y2,Tx)G_2(y1,y2,Tx)G_2(y1,y2,Tx)\}
\]
for all $x$ in $X$ and $y1, y2$ in $Y$, with $x \neq Sy1$, and $x \neq Sy2$, and
\[
G_1^1(Sy1, Sy2, STx) < \frac{1}{2} \max \{G_2(y1,y2,Tx)G_1(x, STx)G_1(x, x, STx),
\]
\[
G_1(x, x, x, STx)G_1(x, x, x, STx)G_1(x, x, x, STx)\}
\]
for all $x$ in $X$ and $y1, y2$ in $Y$, with $x \neq Sy1$, and $x \neq Sy2$. Then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.

**Proof.** Let $\psi : X \to R^+$ defined by $\psi(x) = G_1(x, STx, STx)$ is $G$-continuous on $X$. Since $X$ is compact, there exists a point $u$ in $X$ such that $\psi(u) = G_1(u, STu, STu) = \min \{G_1(x, STx, STx) ; x \in X\}$. Now suppose that $Tu \neq TSTu$. Then $u \neq STu$.
By the inequality (4.4), we have
\[
G_1^1(STu, STu, STSTu) < \frac{1}{2} \max \{G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)G_1(STu, STu, STSTu),
\]
\[
G_1(STu, STu, STu)G_1(STu, STu, STu)G_1(STu, STu, STu)\}
\]
\[
< \frac{1}{2} \max \{G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)G_1(STu, STu, STSTu),0,0\},
\]
\[
G_1(STu, STu, STSTu) < \frac{1}{2} G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)G_1(STu, STu, STSTu),
\]
\[
G_1(STu, STu, STSTu) < \frac{1}{2} G_2(Tu, Tu, TSTu) < G_2(Tu, TSTu, TSTu) < \frac{1}{2} G_2(Tu, TSTu, TSTu)
\]
Using the inequality (4.3), we have
\[
G_2^2(Tu, TSTu, TSTu) < \frac{1}{2} \max \{G_1(u, STu, STu)G_2(Tu, TSTu, TSTu)G_2(Tu, TSTu, TSTu),
\]
\[
G_2(Tu, Tu, Tu)G_1(u, STu, STu)G_2(Tu, Tu, Tu)G_2(Tu, TSTu, TSTu)G_2(Tu, TSTu, TSTu)\}
\]
\[
< \frac{1}{2} \max \{G_1(u, STu, STu)G_2(Tu, TSTu, TSTu)G_2(Tu, TSTu, TSTu),0,0\}
\]
we get $G_2^2(Tu, TSTu, TSTu) < \frac{1}{2} G_1(u, STu, STu)G_2(Tu, TSTu, TSTu)G_2(Tu, TSTu, TSTu),
\]
\[
G_2(Tu, TSTu, TSTu) < \frac{1}{2} G_1(u, STu, STu)
\]
from the inequalities (4.5) and (4.6), we have
\[
\frac{1}{2} G_1(STu, STSTu, STSTu) \leq G_1(STu, STSTu, STSTu) < \frac{1}{2} G_1(u, STu, STu),
\]
\[
G_1(STu, STSTu, STSTu) < G_1(u, STu, STu).
\]
Then $\psi(STu) < \psi(u)$, and this gives us a contradiction, so $TSTu = Tu$. If putting $Tu = w$ and $Sw = z$, then we get $ST(TSTu) = TSTu = Sw = z$, and $w = Tu = TS(Tu) = T(STu) = Tz$. Thus, $Sw = z$ is a fixed point of $ST$ and $Tz = w$ is a fixed point of $TS$. To prove uniqueness, suppose that $ST$ has a second distinct fixed point $z'$. Then applying the inequality (4.4), we have

$$G^3(z, z, z') = G^3(STz, STz, STz') < \frac{1}{2} \max \{G_2(Tz, Tz, Tz'), G_1(z', z', STz') G_1(z, z, STz), G_1(z, z, z') G_1(z', z', STz) \}$$

and it follows that

$$G^3(z, z, z') < \frac{1}{2} G_2(Tz, Tz, Tz') G_1(z', z, z)$$

Applying the inequality (4.3) we have, since $z \neq z' = STz'$,

$$G^3_2(Tz, Tz', Tz') = \frac{1}{2} \max \{G_2(Tz, Tz', Tz'), G_2(Tz', Tz', Tz') \}$$

$$G_2(Tz', Tz', Tz') G_1(z, STz', STz') G_2(Tz', Tz', Tz), G_2(Tz', Tz', Tz') G_2(Tz', Tz', Tz) (4.7)$$

From the inequalities (4.7) and (4.8), we get $G_1(z, z, z') < \frac{1}{2} G_1(z', z', z') \leq G_1(z, z, z')$, This is impossible, and so the fixed point $z$ must be a unique, similar $w$ is a unique fixed point of $TS$.

**Corollary 4.5.** Let $(X, G_1)$ and $(Y, G_2)$ be compact G-metric spaces, and $T$ be a continuous mapping of $X$ into $Y$ and let $S$ be a continuous mapping of $Y$ into $X$ satisfying the inequalities:

$$G^3_2(Tx, TSy_1, TSy_2) < \frac{1}{2} (a_1 G_1(x, Sy_1, Sy_2) G_2(y_1, TSy_1, TSy_2) G_2(y_1, TSy_1, TSy_2) + b_1 G_2(y_1, y_2, Tx) G_1(x, Sy_1, Sy_2) G_2(y_1, y_2, Tx) + c_1 G_1(y_1, TSy_1, TSy_2) G_2(y_1, y_2, Tx) G_2(y_1, y_2, Tx))$$

for all $x$ in $X$ and $y_1, y_2$ in $Y$, with $x \neq Sy_1$, and $x \neq Sy_2$, and

$$G^3_1(Sy_1, Sy_2, STx) < \frac{1}{2} (a_2 G_2(y_1, y_2, Tx) G_1(x, x, STx) G_1(x, x, STx) + b_2 G_1(x, Sy_1, Sy_2) G_2(y_1, y_2, Tx) G_1(x, Sy_1, Sy_2) + c_2 G_1(x, x, STx) G_1(x, Sy_1, Sy_2) G_1(x, Sy_1, Sy_2))$$

for all $x$ in $X$ and $y_1, y_2$ in $Y$, with $y_1 \neq Ty, y_2 \neq Ty$ and $a_1, a_2, b_1, b_2, c_1, c_2 \in R^+$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$. Then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.

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