The Direct Theorem of Baskakov-Bézier-type Operators in Hölder Space *

Qiulan Qi1,2,† Wenzhe Li2
1 School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, 050024, P.R. China.
2 Hebei Key Laboratory of Computational Mathematics and Applications, Shijiazhuang, 050024, P.R. China

Abstract:
In this paper, the applications of Baskakov-Bézier and Baskakov-Kantorovich-Bézier operators in Hölder space were discussed. Using the equivalent relation between the modulus of continuity and the K-functional, the direct approximation of continuous functions in Hölder space by Baskakov-Bézier-type operators was obtained.

Keywords: Baskakov-Bézier operators, Baskakov-Kantorovich-Bézier operators, Hölder space, Modulus of continuity, K-functional

MSC: 41A30, 41A36.

1. INTRODUCTION

Recently, there are many results on the convergence speed of different approximation processes in the Hölder space[1–6]. In Computer Aided Geometric Design, Bézier basis functions are very useful and their analytical properties have been studied by many authors[7–11]. In this paper, we will study the approximation properties of Baskakov-Bézier-type operators in the Hölder space.

First we introduce two semi-norms: Let $0 < \delta \leq 1$,

$$\theta_\alpha(f, \delta) = \sup_{0 < |x-y| \leq \delta} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \quad \theta_\alpha(f) = \sup_{0 < \delta \leq 1} \theta_\alpha(f, \delta).$$

Due to the definition of the classical modulus of continuity $\omega_\alpha(f, \delta) = \sup \{|f(x) - f(y)| : |x-y| \leq \delta\}$, when $0 < \delta \leq 1$, one has

$$\theta_\alpha(f, \delta) = \sup_{0 < t \leq \delta} \frac{\omega_{\alpha}(f, t)}{t^\alpha}. \quad (1)$$

To describe the results, we need to introduce some symbols. Let $0 \leq \alpha \leq 1$,

$$C^{1,\alpha}[0, \infty) = \{f \in C^1[0, \infty) : \|f\|_{1,\alpha} : \frac{1}{\alpha} \sum_{k=0}^{\infty} \|f^{(k)}\| + 1 \|f'|_\alpha < \infty\},$$

$$L_{\infty}^{1,\alpha}(0, \infty) = \{f \in L_\infty^1[0, \infty) : \|f\|_{1,\alpha} : \frac{1}{\alpha} \sum_{k=0}^{\infty} \|f^{(k)}\| + 1 \|f'|_\alpha < \infty\},$$

$$1 |g|_\alpha := \sup_{h>0} h^{-\alpha} \max_{x \in [0,\infty]} |g(x + h) - g(x)| \sim \theta_\alpha(f).$$

Due to the equivalence of $\omega_{\alpha}(f, \delta)$ and K-functional, we will use the following semi-norm

$$2 |g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot K(g, h; C, C^1),$$

$$K(g, h) := K(g, h; C, C^1) := \inf_{G \in C^{1,\alpha}(0, \infty)} \{\|g - G\| + h\|G'\|\},$$

obviously $1 |g|_\alpha \sim 2 |g|_\alpha$. The Baskakov-Bézier-type operators are defined by

$$V_{n,\gamma}(f, x) = \sum_{k=0}^{\infty} \int_{k/n}^{(k+1)/n} f(u) du J_{n,k}(x) - J_{n,k+1}(x),$$

$$V_{n,\gamma}^*(f, x) = \sum_{k=0}^{\infty} n \int_{k/n}^{(k+1)/n} f(u) du J_{n,k}(x) - J_{n,k+1}(x),$$

where $\gamma \geq 1$, $J_{n,k}(x) = \sum_{v=1}^{\infty} v_{n,\gamma}(x)$, $v_{n,\gamma}(x) = \left(\frac{n+\gamma-1}{x(1+x)}\right)^{-1/\gamma}.$

The main results of this paper are

Theorem 1.1 Let $0 \leq \beta \leq \alpha \leq 1, \gamma \geq 1$ for $f \in C^{1,\alpha}[0, \infty)$, one has

$$\|V_{n,\gamma} f - f\|_{1,\beta} \leq C \cdot (n-1) \frac{2^{\alpha}}{\alpha} \cdot \|f\|_{1,\alpha}.$$  

Theorem 1.2 Let $0 \leq \beta \leq \alpha \leq 1, \gamma \geq 1$ for $f \in L_{\infty}^{1,\alpha}(0, \infty)$, one has

$$\|V_{n,\gamma}^* f - f\|_{1,\beta} \leq C \cdot (n-1) \frac{2^{\alpha}}{\alpha} \cdot \|f\|_{1,\alpha}.$$  

Remark 1.3 Throughout this paper, C denotes a positive constant independent of $n$ and $x$ and not necessarily the same at each occurrence. We will only prove Theorem 1.1, we can use the same methods to get Theorem 1.2, the details will be omitted.

---

*This work is partially supported by NSF of China(11571089,11871191), NSF of Hebei Province(ZD2019053).
†Correspondence author. E-mail: qiquilan@163.com
2. AUXILIARY RESULTS
First we list some basic properties which will be used in next sections.

Lemma 2.1 If \( \omega_1(\gamma, f, t) \leq C \omega_1(f, t) \), one has \( \omega_1(V_{n, \gamma} f, t) \leq C \omega_1(f, t) \).

Proof. According to the relationship between the modulus of continuity and the K-functional, we have

\[
\omega_1(V_{n, \gamma} f, t) \leq C \inf_{g \in C^1} \{ ||V_{n, \gamma} f - g|| + t ||g|| \}
\]

\[
\leq C \inf_{g \in C^1} \{ ||V_{n, \gamma} f - V_{n, \gamma} g|| + t ||V_{n, \gamma} g|| \}
\]

\[
\leq C \inf_{g \in C^1} \{ ||f - g|| + t ||g|| \} \leq C \omega_1(f, t).
\]

Lemma 2.2 Let \( t \geq 0 \), one has

\[
\omega_1(V_{n, \gamma} f, t) \leq C \omega_1(f, t).
\]

Proof. According to the relationship between the modulus of continuity and the K-functional, we have

\[
\omega_1(V_{n, \gamma} f, t) \leq C \inf_{g \in C^1} \{ ||V_{n, \gamma} f - g|| + t ||g|| \}
\]

\[
\leq C \inf_{g \in C^1} \{ ||V_{n, \gamma} f - V_{n, \gamma} g|| + t ||V_{n, \gamma} g|| \}
\]

\[
\leq C \inf_{g \in C^1} \{ ||f - g|| + t ||g|| \} \leq C \omega_1(f, t).
\]

Lemma 2.3 If \( f \in C[0, \infty) \), one has \( ||f - V_{n, \gamma} f|| \leq C \omega_1(f, 1/n) \).

If \( f \in L_\infty[0, \infty) \), one has \( ||f - V_{n, \gamma} f|| \leq C \omega_1(f, 1/n) \).

Lemma 2.4 If \( f \in C^{0, \alpha}[0, \infty) \), one has

\[
\max_{x \in [0, \infty)} |\Delta_h V_{n, \gamma}(f, x)| \leq h^{\alpha - 1} |f|_\alpha.
\]

If \( f \in L_\infty^{0, \alpha}[0, \infty) \), one has \( \max_{x \in [0, \infty)} |\Delta_h V_{n, \gamma}^*(f, x)| \leq h^{\alpha - 1} |f|_\alpha \).

Proof. Using the following relation

\[
\max_{x \in [0, \infty)} |\Delta_h V_{n, \gamma} f(x)| = \omega_1(V_{n, \gamma} f, h) \leq C \omega_1(f, h) = Ch^{\alpha - 1} |f|_\alpha,
\]

one can get (2).

Lemma 2.5 If \( f \in C^1[0, \infty) \), one has

\[
||(f - V_{n, \gamma} f)'|| \leq ||f' - V_{n+1, \gamma} f'|| + \gamma \omega_1(f', 1/n) + \frac{\gamma(\gamma - 1)}{n} ||f'||.
\]

If \( f' \in L_\infty[0, \infty) \), one has

\[
||(f - V_{n, \gamma} f)'|| \leq ||f' - V_{n+1, \gamma} f'|| + \gamma \omega_1(f', 1/n) + \frac{\gamma(\gamma - 1)}{n} ||f'||.
\]

Proof. Applying the properties of \( J_{n, k}(x) \) in Lemma 2.1, we have

\[
V_{n, \gamma}(f, x) = \sum_{k=0}^{\infty} f(k/n) \gamma \left( J_{n, k}^{\gamma - 1}(x) - J_{n, k+1}^{\gamma - 1}(x) \right)
\]

\[
= f(0) J_{n, 0}^{\gamma - 1}(x) J'_{n, 0}(x) + \sum_{k=0}^{\infty} \gamma \left( f(k/n + 1/n) - f(k/n) \right) \cdot J_{n, k+1}^{\gamma - 1}(x) J'_{n, k+1}(x)
\]

\[
= \sum_{k=0}^{\infty} f'(\xi_k) \cdot \gamma \cdot J_{n, k+1}^{\gamma} \cdot v_{n+1, k}(x), \quad for \quad \xi_k \in \left( \frac{k}{n}, \frac{k + 1}{n} \right).
\]

Noting that

\[
V_{n+1, \gamma}(f', x) = \sum_{k=0}^{\infty} f'(k/n + 1) \cdot J_{n+1, k}^{\gamma}(x) - J_{n+1, k+1}^{\gamma}(x)
\]

\[
= \sum_{k=0}^{\infty} f'(k/n + 1) \cdot \gamma \cdot g_{k}(x) \cdot v_{n+1, k}(x), \quad for \quad g_k(x) \in \left( J_{n+1, k+1}(x), J_{n+1, k}(x) \right),
\]
we can write

\[ V'_{n,\gamma}(f, x) = V_{n+1,\gamma}(f', x) + \gamma \sum_{k=0}^{\infty} \left[ f'(\xi_k)J_{n,k+1}^{-1}(x) - f'\left(\frac{k}{n+1}\right)g_k^{-1}(x)\right]v_{n+1,k}(x) \]

\[ = V_{n+1,\gamma}(f', x) + \gamma \sum_{k=0}^{\infty} \left[ f'(\xi_k)J_{n,k+1}^{-1}(x) - f'\left(\frac{k}{n+1}\right)J_{n,k+1}^{-1}(x) \right] \\
+ f'\left(\frac{k}{n+1}\right)J_{n,k+1}^{-1}(x) - f'\left(\frac{k}{n+1}\right)g_k^{-1}(x)\right]v_{n+1,k}(x) \]

\[ = V_{n+1,\gamma}(f', x) + \gamma \sum_{k=0}^{\infty} \left[ f'(\xi_k) - f'\left(\frac{k}{n+1}\right)\right]J_{n,k+1}^{-1}(x)v_{n+1,k}(x) \]

\[ + \gamma \sum_{k=0}^{\infty} f'\left(\frac{k}{n+1}\right)\left[J_{n,k+1}^{-1}(x) - g_k^{-1}(x)\right]v_{n+1,k}(x). \]

Hence

\[ \| (f - V_{n,\gamma}f') \| \leq \| f' - V_{n+1,\gamma}f' \| + \gamma \omega_{1}(f, \frac{1}{n}) + \frac{\gamma(\gamma - 1)}{n} \| f' \|. \]

**Remark 2.6** By the Lemma 2.4, Lemma 2.3 and (1), one has

\[ \| (f' - V_{n,\gamma}f') \| \leq C \cdot (n - 1)^{-\frac{2}{n}} \theta_n(f') + \frac{\gamma}{n^\alpha} \theta_n(f') + \frac{\gamma(\gamma - 1)}{n} \| f' \|. \]

\[ \sum_{k=0}^{1} \| (f^{(k)} - V_{n,\gamma}^{(k)}f) \| \leq C \cdot n^{-\frac{2}{n}} \theta_n(f) + C \cdot (n - 1)^{-\frac{2}{n}} \theta_n(f') + \frac{\gamma}{n^\alpha} \theta_n(f') + \frac{\gamma(\gamma - 1)}{n} \| f' \|. \]

3 PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1.

If \( f' \in Lip_{1,\alpha} \Rightarrow V_{n,\gamma}f' \in Lip_{1,\alpha} \), we have \( \theta_n(V_{n,\gamma}f') \leq \theta_n(f') \). For \( 0 \leq h \leq \frac{1}{\sqrt{n}} \), using (5), we get

\[ \sup_{0 < h \leq \frac{1}{\sqrt{n}}} h^{-\beta} \max_{x \in [0, \infty)} |\Delta_h(f' - V_{n,\gamma}f, x)| \leq n^{\frac{\alpha-\alpha}{2}} \sup_{0 < h \leq \frac{1}{\sqrt{n}}} h^{-\alpha} \max_{x \in [0, \infty)} \{ |\Delta_hf'(x)| + |\Delta_hV_{n,\gamma}f, f(x)| \} \]

\[ \leq n^{\frac{\alpha-\alpha}{2}} \theta_n(f') + \theta_n(V_{n,\gamma}f, f(x)) \leq 2n^{\frac{\alpha-\alpha}{2}} \theta_n(f'). \]

For \( \frac{1}{\sqrt{n}} \leq h \leq 1 \), applying (4), we have

\[ \sup_{\frac{1}{\sqrt{n}} \leq h \leq 1} h^{-\beta} \max_{x \in [0, \infty)} |\Delta_h(f' - V_{n,\gamma}f, x)| \leq n^{\frac{\alpha}{2}} \| f'(x) - V_{n,\gamma}f, f(x) \| \]

\[ \leq n^{\frac{\alpha}{2}} \cdot C \cdot (n - 1)^{-\frac{2}{n}} \theta_n(f') + \frac{\gamma}{n^\alpha} \theta_n(f') + \frac{\gamma(\gamma - 1)}{n} \| f' \| \]

\[ \leq C \cdot n^{\frac{\alpha-\alpha}{2}} \left( \left(\frac{1}{n - 1}\right)^{\frac{2}{n}} + \frac{\gamma}{n^\frac{\alpha}{2}} \theta_n(f') + \frac{\gamma(\gamma - 1)}{n^{1-\frac{2}{n}}} \| f' \| \right) \leq C \cdot n^{\frac{\alpha-\alpha}{2}} \cdot (\theta_n(f') + \| f' \|). \]

This completes the proof of the theorem.
Acknowledgments
We express our gratitude to the referees for their helpful suggestions.

References