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The Direct Theorem of Baskakov-Bézier-type Operators in Hölder Space *

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Abstract:
In this paper, the applications of Baskakov-Bézier and Baskakov-Kantorovich-Bézier operators in Hölder space were discussed. Using the equivalent relation between the modulus of continuity and the K-functional, the direct approximation of continuous functions in Hölder space by Baskakov-Bézier-type operators was obtained.

Keywords: Baskakov-Bézier operators, Baskakov-Kantorovich -Bézier operators, Hölder space, Modulus of continuity, K-functional

MSC: 41A30, 41A36.

1. INTRODUCTION

Recently, there are many results on the convergence speed of different approximation processes in the Hölder space[1−6]. In Computer Aided Geometric Design, Bézier basis functions are very useful and their analytical properties have been studied by many authors[7−11]. In this paper, we will study the approximation properties of Baskakov-Bézier-type operators in the Hölder space.

First we introduce two semi-norms: Let $0 < \delta \le 1$,

$$\theta_\alpha(f,\delta) = \sup_{0<|x-y|\le\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \quad \theta_\alpha(f) = \sup_{0<\delta\le 1} \theta_\alpha(f,\delta).$$

Due to the definition of the classical modulus of continuity $\omega_1(f,\delta) = \sup\{|f(x)-f(y)| : |x-y| \le \delta\}$, when $0 < \delta \le 1$, one has

$$\theta_\alpha(f,\delta) = \sup_{0<t\le\delta} \frac{\omega_1(f,t)}{t^\alpha}. \qquad (1)$$

To describe the results, we need to introduce some symbols. Let $0 \le \alpha \le 1$,

$$C^{1,\alpha}[0,\infty) : = \{f \in C^1[0,\infty) : \|f\|_{1,\alpha} : = \sum_{k=0}^{1} \|f^{(k)}\| +^1 |f'|_\alpha < \infty\},$$

$$L^{1,\alpha}_\infty[0,\infty) : = \{f \in L^1_\infty[0,\infty) : \|f\|_{1,\alpha} : = \sum_{k=0}^{1} \|f^{(k)}\| +^1 |f'|_\alpha < \infty\},$$

$${}^1|g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot \max_{x\in[0,\infty)} |g(x+h) - g(x)| \sim \theta_\alpha(f).$$

Due to the equivalence of $\omega_1(f,h)$ and K-functional, we will use the following semi-norm

$${}^2|g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot K(g,h;C,C^1),$$

$$K(g,h) := K(g,h;C,C^1) := \inf_{G\in C^1[0,\infty)} \{\|g-G\| + h\|G'\|\},$$

obviously ${}^1|g|_\alpha \sim^2 |g|_\alpha$. The Baskakov-Bézier-type operators are defined by

$$V_{n,\gamma}(f,x) = \sum_{k=0}^{\infty} f(\frac{k}{n})[J^\gamma_{n,k}(x) - J^\gamma_{n,k+1}(x)],$$

$$V^*_{n,\gamma}(f,x) = \sum_{k=0}^{\infty} n \int_{k/n}^{(k+1)/n} f(u)du[J^\gamma_{n,k}(x) - J^\gamma_{n,k+1}(x)],$$

where $\gamma \ge 1$, $J_{n,k}(x) = \sum_{i=k}^\infty v_{n,i}(x)$, $v_{n,i}(x) = \binom{n+i-1}{i} x^i (1+x)^{-n-i}$.

The main results of this paper are

Theorem 1.1 Let $0 \le \beta \le \alpha \le 1, \gamma \ge 1$ for $f \in C^{1,\alpha}[0,\infty)$, one has

$$\|V_{n,\gamma}f - f\|_{1,\beta} \le C \cdot (n-1)^{\frac{\beta-\alpha}{2}} \cdot \|f\|_{1,\alpha}.$$

Theorem 1.2 Let $0 \le \beta \le \alpha \le 1, \gamma \ge 1$ for $f \in L^{1,\alpha}_\infty[0,\infty)$, one has

$$\|V^*_{n,\gamma}f - f\|_{1,\beta} \le C \cdot (n-1)^{\frac{\beta-\alpha}{2}} \cdot \|f\|_{1,\alpha}.$$

Remark 1.3 Throughout this paper, C denotes a positive constant independent of n and x and not necessarily the same at each occurrence. We will only prove Theorem 1.1, we can use the same methods to get Theorem 1.2, the details will be omitted.

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2. AUXILIARY RESULTS

First we list some basic properties which will be used in next sections.

Lemma 2.1[11]

$$1 = J_{n,0}(x) > J_{n,1}(x) > \cdots > J_{n,k}(x) > J_{n,k+1}(x) > \cdots;$$
$$0 < J^{\gamma}_{n,k}(x) - J^{\gamma}_{n,k+1}(x) < \gamma v_{n,k}(x), \quad \gamma \geq 1;$$
$$J'_{n,0}(x) = 0, \quad J'_{n,k}(x) = n v_{n+1,k-1}(x) > 0 \ \ (k = 1, 2, \cdots).$$

Lemma 2.2 Let $t \geq 0$, one has

$$\omega_1(V_{n,\gamma}f, t) \leq C\omega_1(f, t), \ \ \omega_1(V^*_{n,\gamma}f, t) \leq C\omega_1(f, t).$$

Proof. According to the relationship between the modulus of continuity and the K - functional[12]

$$\begin{aligned} & \omega_1(V_{n,\gamma}f, t) \leq C \inf_{g \in C^{1,\alpha}} \{\|V_{n,\gamma}f - g\| + t\|g\|\} \\ \leq & \ C \inf_{g \in C^{1,\alpha}} \{\|V_{n,\gamma}f - V_{n,\gamma}g\| + t\|V_{n,\gamma}g\|\} \\ \leq & \ C \inf_{g \in C^{1,\alpha}} \{\|f - g\| + t\|g\|\} \leq C\omega_1(f, t). \end{aligned}$$

Lemma 2.3[11] If $f \in C[0, \infty)$, one has $\|f - V_{n,\gamma}f\| \leq C \cdot \omega_1(f, \frac{1}{\sqrt{n}})$.
If $f \in L_\infty[0, \infty)$, one has $\|f - V^*_{n,\gamma}f\| \leq C \cdot \omega_1(f, \frac{1}{\sqrt{n}})$.

Lemma 2.4 If $f \in C^{0,\alpha}[0, \infty)$, one has

$$\max_{x \in [0,\infty)} |\Delta_h V_{n,\gamma}(f, x)| \leq h^{\alpha} \cdot^{1} |f|_{\alpha}. \tag{2}$$

If $f \in L^{0,\alpha}_{\infty}[0, \infty)$, one has $\max_{x \in [0,\infty)} |\Delta_h V^*_{n,\gamma}(f, x)| \leq h^{\alpha} \cdot^{1} |f|_{\alpha}$.

Proof. Using the following relation

$$\max_{x \in [0,\infty)} |\Delta_h V_{n,\gamma}f(x)| = \omega_1(V_{n,\gamma}f, h) \leq C\omega_1(f, h) = Ch^{\alpha} \cdot^{1} |f|_{\alpha},$$

one can get (2).

Lemma 2.5 If $f \in C^1[0, \infty)$, one has

$$\|(f - V_{n,\gamma}f)'\| \leq \|f' - V_{n+1,\gamma}f'\| + \gamma\omega_1(f', \frac{1}{n}) + \frac{\gamma(\gamma - 1)}{n}\|f'\|. \tag{3}$$

If $f' \in L_\infty[0, \infty)$, one has

$$\|(f - V^*_{n,\gamma}f)'\| \leq \|f' - V^*_{n+1,\gamma}f'\| + \gamma\omega_1(f', \frac{1}{n}) + \frac{\gamma(\gamma - 1)}{n}\|f'\|.$$

Proof. Applying the properties of $J_{n,k}(x)$ in Lemma 2.1, we have

$$\begin{aligned} & V'_{n,\gamma}(f, x) = \sum_{k=0}^{\infty} f(\frac{k}{n})\gamma \left\{ J^{\gamma-1}_{n,k}(x)J'_{n,k}(x) - J^{\gamma-1}_{n,k+1}(x)J'_{n,k+1}(x) \right\} \\ = & \ f(0)\gamma J^{\gamma-1}_{n,0}(x)J'_{n,0}(x) + \sum_{k=0}^{\infty} \gamma \left[f(\frac{k+1}{n}) - f(\frac{k}{n}) \right] \cdot J^{\gamma-1}_{n,k+1}(x)J'_{n,k+1}(x) \\ = & \ \sum_{k=0}^{\infty} f'(\xi_k) \cdot \gamma \cdot J^{\gamma-1}_{n,k+1}(x) \cdot v_{n+1,k}(x), \quad for \quad \xi_k \in (\frac{k}{n}, \frac{k+1}{n}). \end{aligned}$$

Noting that

$$V_{n+1,\gamma}(f', x) = \sum_{k=0}^{\infty} f'(\frac{k}{n+1}) \cdot [J^{\gamma}_{n+1,k}(x) - J^{\gamma}_{n+1,k+1}(x)]$$

$$= \sum_{k=0}^{\infty} f'(\frac{k}{n+1}) \cdot \gamma \cdot g^{\gamma-1}_{k}(x) \cdot v_{n+1,k}(x), \quad for \quad g_k(x) \in (J_{n+1,k+1}(x), J_{n+1,k}(x)),$$

we can write

$$
\begin{aligned}
V'_{n,\gamma}(f,x) &= V_{n+1,\gamma}(f',x) + \gamma\cdot\sum_{k=0}^{\infty}[f'(\xi_k)J_{n,k+1}^{\gamma-1}(x) - f'(\frac{k}{n+1})g_k^{\gamma-1}(x)]v_{n+1,k}(x)\\
&= V_{n+1,\gamma}(f',x) + \gamma\sum_{k=0}^{\infty}[f'(\xi_k)J_{n,k+1}^{\gamma-1}(x) - f'(\frac{k}{n+1})J_{n,k+1}^{\gamma-1}(x)\\
&\quad + f'(\frac{k}{n+1})J_{n,k+1}^{\gamma-1}(x) - f'(\frac{k}{n+1})g_k^{\gamma-1}(x)]v_{n+1,k}(x)\\
&= V_{n+1,\gamma}(f',x) + \gamma\sum_{k=0}^{\infty}[f'(\xi_k) - f'(\frac{k}{n+1})]J_{n,k+1}^{\gamma-1}(x)v_{n+1,k}(x)\\
&\quad + \gamma\sum_{k=0}^{\infty}f'(\frac{k}{n+1})[J_{n,k+1}^{\gamma-1}(x) - g_k^{\gamma-1}(x)]v_{n+1,k}(x).
\end{aligned}
$$

Hence

$$\|(f - V_{n,\gamma}f)'\| \le \|f' - V_{n+1,\gamma}f'\| + \gamma\omega_1(f', \frac{1}{n}) + \frac{\gamma(\gamma-1)}{n}\|f'\|.$$

Remark 2.6 By the Lemma 2.4, Lemma 2.3 and (1), one has

$$\|(f' - V'_{n,\gamma}f)\| \le C\cdot(n-1)^{-\frac{\alpha}{2}}\theta_\alpha(f') + \frac{\gamma}{n^\alpha}\theta_\alpha(f') + \frac{\gamma(\gamma-1)}{n}\|f'\|. \tag{4}$$

$$\sum_{k=0}^{1}\|(f^{(k)} - V_{n,\gamma}^{(k)}f)\| \le C\cdot n^{-\frac{\alpha}{2}}\theta_\alpha(f) + C\cdot(n-1)^{-\frac{\alpha}{2}}\theta_\alpha(f') + \frac{\gamma}{n^\alpha}\theta_\alpha(f') + \frac{\gamma(\gamma-1)}{n}\|f'\|. \tag{5}$$

3 PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1.

If $f' \in Lip_L\alpha \Rightarrow V'_{n,\gamma}f \in Lip_L\alpha$, we have $\theta_\alpha(V'_{n,\gamma}f) \le \theta_\alpha(f')$. For $0 \le h \le \frac{1}{\sqrt{n}}$, using (5), we get

$$
\begin{aligned}
&\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\beta}\cdot\max_{x\in[0,\infty)}|\Delta_h(f' - V'_{n,\gamma}(f,x)|\\
&\le n^{\frac{\beta-\alpha}{2}}\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\alpha}\max_{x\in[0,\infty)}\{|\Delta_h f'(x)| + |\Delta_h V'_{n,\gamma}(f,x)|\}\\
&\le n^{\frac{\beta-\alpha}{2}}[\theta_\alpha(f') + \theta_\alpha(V'_{n,\gamma}(f,x))] \le 2n^{\frac{\beta-\alpha}{2}}\theta_\alpha(f').
\end{aligned}
$$

For $\frac{1}{\sqrt{n}} \le h \le 1$, applying (4), we have

$$
\begin{aligned}
&\sup_{\frac{1}{\sqrt{n}}\le h\le 1} h^{-\beta}\cdot\max_{x\in[0,\infty)}|\Delta_h(f' - V'_{n,\gamma}(f,x)| \le n^{\frac{\beta}{2}}\|f'(x) - V'_{n,\gamma}(f,x)\|\\
&\le n^{\frac{\beta}{2}}\cdot[C\cdot(n-1)^{-\frac{\alpha}{2}}\theta_\alpha(f') + \frac{\gamma}{n^\alpha}\theta_\alpha(f') + \frac{\gamma(\gamma-1)}{n}\|f'\|]\\
&\le C\cdot n^{\frac{\beta-\alpha}{2}}\{[(\frac{n}{n-1})^{\frac{\alpha}{2}} + \frac{\gamma}{n^{\frac{\alpha}{2}}}]\theta_\alpha(f') + \frac{\gamma(\gamma-1)}{n^{1-\frac{\alpha}{2}}}\|f'\|\} \le C\cdot n^{\frac{\beta-\alpha}{2}}\cdot(\theta_\alpha(f') + \|f'\|)\,.
\end{aligned}
$$

This completes the proof of the theorem.

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