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Adaptive Step-size Nonlinear Explicit Integration Algorithm for ODEs

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Abstract

In this paper, an adaptive step-size nonlinear explicit integration algorithm for solving ordinary differential equations is present. The algorithm is based on a new high order nonlinear explicit integration scheme. The order of convergence and stability properties are investigated and shown to be of fifth-order and large stability region. Several test problems are solved and the numerical results are presented. The results confirm that the algorithm is effective and appropriate for solving stiff and singularly perturbed ordinary differential equations.

Keywords: Numerical integration; Nonlinear schemes, Adaptive techniques, Stiff and singularly perturbed ODEs.

I. INTRODUCTION

Ordinary differential equations (ODE) are widely used to describe continuous time physical problems. In most of the cases, these problems are too complicated to solve analytically. Alternatively, the numerical methods can provide approximate solutions rather than the analytic solution of problems. Stiff and singularly perturbed differential equations are ones of the most interesting ODEs which frequently arise in applied science and engineering [1-12]. Applying classical numerical methods for solving these types of problems requires very small step-size to overcome stability restriction while applying standard implicit methods requires a large number of iterations for convergence within each time step. In this paper, an adaptive step-size nonlinear explicit integration algorithm for solving ordinary differential equations is present. The algorithm is based on a new high order nonlinear explicit integration scheme. The order of convergence and stability properties are investigated and shown to be of fifth-order and large stability region. Several test problems are solved and the numerical results are presented. The results confirm that the algorithm is effective and appropriate for solving stiff and singularly perturbed ordinary differential equations.

II. NONLINEAR EXPLICIT SCHEME

Consider the IVP

$$y' = \phi(x, y), \;\; y(a) = y_0 \;, y, \phi(x, y) \in \mathbb{R}, \tag{1}$$

where ϕ is assumed to satisfy all the requirements in order that (1) has a unique solution. The interval $[a,b]$ is divided into a number of subintervals $[x_j, x_{j+1}]$ with $x_0 = a$ and $x_j = x_0 + jh$, where h is the step size , $j = 1, 2, \ldots.$, and $x_j \le b$. Following the idea presented in [8, 10, 11] we consider Taylor's expansions of y_{j+1} and y_{j-1} about x_j as follows

$$\Delta_{j+1} \equiv y_{j+1} - y_j = hy'_j + \frac{h^2}{2} y''_j + \frac{h^3}{6} y'''_j + \frac{h^4}{24} y_j^{(4)} + \frac{h^5}{120} y_j^{(5)} + \frac{h^6}{720} y_j^{(6)} + O(h^7) \tag{2}$$

$$\Delta_j \equiv y_j - y_{j-1} = hy'_j - \frac{h^2}{2} y''_j + \frac{h^3}{6} y'''_j - \frac{h^4}{24} y_j^{(4)} + \frac{h^5}{120} y_j^{(5)} - \frac{kh^6}{720} y_j^{(6)} + O(h^7) \tag{3}$$

where k is a positive constant which will be determined later from the stability analysis.

From Eq. (2) and Eq. (3) we have

$$\Delta_{j+1}\Delta_j \equiv \Delta_{j+1}\left(hy'_j - \frac{h^2}{2} y''_j + \frac{h^3}{6} y'''_j - \frac{h^4}{24} y_j^{(4)} + \frac{h^5}{120} y_j^{(5)} - \frac{h^6}{720} y_j^{(6)} + O(h^7) \right)$$

then

$$\Delta_{j+1} \cong \frac{2h\left(360\,\theta_1 + 30\,h^2\theta_2 + h^4\,\theta_3\right) + O(h^6)}{\left(720y'_j - 360hy''_j + 120h^2 y'''_j - 30h^3 y_j^{(4)} + 6h^4 y_j^{(5)} - kh^5 y_j^{(6)} + O(h^6)\right)} \tag{4}$$

where

$$\theta_1 = \left(y'_j\right)^2, \;\; \theta_2 = 4y'_j y'''_j - 3\left(y''_j\right)^2, \;\; \theta_3 = 6y'_j y_j^{(5)} - 15y''_j y_j^{(4)} + 10\left(y'''_j\right)^2 .$$

From (4) the numerical scheme is readily obtained, which may be written in the form

$$y_{j+1} \cong y_j + \frac{2h\left(360\,\theta_1 + 30\,h^2\theta_2 + h^4\,\theta_3\right)}{\left(720y'_j - 360hy''_j + 120h^2 y'''_j - 30h^3 y_j^{(4)} + 6h^4 y_j^{(5)} - kh^5 y_j^{(6)}\right)} \tag{5}$$

where $y'_j = \phi(x_j, y_j)$ and the higher derivatives can be obtained from (1) by successive differentiation as follows

$$y''_j = \phi''(x_j, y_j) = \frac{\partial \phi}{\partial x}(x_j, y_j) + \frac{\partial \phi}{\partial y}(x_j, y_j) y'_j \ ,$$

$$y'''_j = \phi'''(x_j, y_j) = \frac{\partial \phi''}{\partial x}(x_j, y_j) + \frac{\partial \phi''}{\partial y}(x_j, y_j) y'_j \quad , \ \ldots\ldots \ \ldots\ldots ,$$

$$y_j^{(6)} = \phi^{(6)}(x_j, y_j) \ .$$

II.I. STABILITY ANALYSIS

In order to examine the present method for the stability, let us consider the differential equation,

$$y' = \lambda y . \tag{6}$$

where λ is a complex constant and $Re(\lambda) < 0$ .For this equation, Eq.(5) can be rewritten as

$$y_{j+1} = \frac{720 + 360(h\lambda) + 120(h\lambda)^2 + 30(h\lambda)^3 + 6(h\lambda)^4 - (2-k)(h\lambda)^5}{720 - 360(h\lambda) + 120(h\lambda)^2 - 30(h\lambda)^3 + 6(h\lambda)^4 - k\,(h\lambda)^5} y_j$$

Setting $z = \lambda y$ in the above equation, the amplification factor is therefore

$$R(z) = \frac{720 + 360z + 120z^2 + 30z^3 + 6z^4 - (2-k)z^5}{720 - 360z + 120z^2 - 30z^3 + 6z^4 - kz^5} \tag{7}$$

For $k = 2$, we have

$$R(z) = \frac{720 + 360z + 120z^2 + 30z^3 + 6z^4}{720 - 360z + 120z^2 - 30z^3 + 6z^4 - 2z^5} \tag{8}$$

which has larger stability region as shown in Figure 1

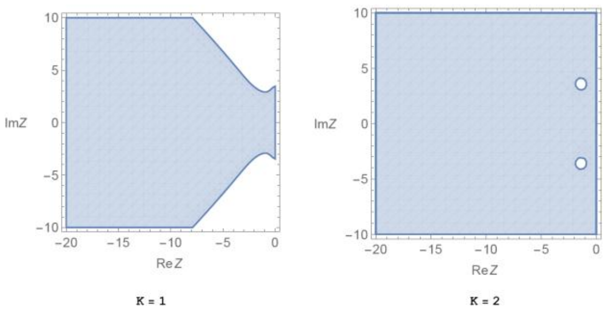

Figure 1. Stability regions for $k = 1$ (left) and $k = 2$ (right)

From (8) , the two-term recurrence relation (5) becomes

$$y_{j+1} \cong y_j + \frac{h\left(360\,\theta_1 + 30\,h^2\,\theta_2 + h^4\,\theta_3\right)}{\left(360 y'_j - 1800 h y''_j + 60 h^2 y'''_j - 15 h^3 y_j^{(4)} + 3 h^4 y_j^{(5)} - h^5 y_j^{(6)}\right)} \ , \tag{9}$$

III. LOCAL TRUNCATION ERROR

The local truncation error τ is readily obtained from subtracting Eq. (9) from Taylor series Eq. (2), which may be written in the form

$$\tau = \left(\frac{y_j^6}{720}\right) h^6 - O(h^7). \tag{10}$$

This means that the method described in Eq. (2) through Eq. (10) has a fifth order of accuracy. Consequently the numerical integration of the initial-value problem (1) is L-stable and convergent with order 5.

Remark. The local truncation error of the method can be written as a function of the constant k as follows

$$\tau = \left(\frac{(k-1) y_j^6}{720}\right) h^6 - O(h^7) \quad , \tag{11}$$

which means that the method is A-stable and has at least six order of accuracy at $k = 1$. In this case the local truncation error can be written as

$$\tau = \left(\frac{4 y'_j y_j^7 - 28 y''_j y_j^6 + 56 y'''_j y_j^5 - 35 (y_j^{(4)})^2}{20160 y'_j}\right) h^7 + O(h^8). , \tag{12}$$

IV. ADAPTIVE STEP-SIZE ALGORITHM

The discrete solution of the IVP (1) is obtained using the fifth order method over a non uniform step size by involving a monitor function based on the estimated error in Eq. (10).

Assume that the results by present method in discrete solution with a specific error tolerance Tol .

$$\left| y(x_{j+1}) - y_{j+1} \right| \leq Tol \quad , \tag{13}$$

From Eq. (13) and Eq. (10) we have an optimum step size $h_j = x_{j+1} - x_j$ verifies

$$\left(\frac{y_j^6}{720}\right) h_j^6 \leq Tol \quad , \tag{14}$$

Thus

$$h_j \leq \left(\frac{720 Tol}{y_j^6}\right)^{1/6} \quad . \tag{15}$$

It is possible that for certain combination of the values appearing in the denominator in (9), this denominator vanishes. In that case, it is necessary to modify slightly the step size on this step and take $h_j = m h_j$ instead of h_j , where the factor m is taken nearly unity, say $m = 0.9$. The same case may appears in the denominator in (15). In that case, we take $h_j = h_{\max}$, where $h_{\max}$ is the maximum allowed step size. In our algorithm we set $h_{\max} = 0.02$

These details will be combined in the following algorithm:

Algorithm steps:

Step I: $input\ \ a, b, y(a), Tol =, h_{\max}(max\ allowed\ step\ size\ )$

$Set\ \ x_0 = a,\ \ index = 1,\ \ y_0 = y(a), Y(1) = y_0\ \ , x_0 = a$

Step II: obtain the successive derivatives $\phi'(x, y), \phi''(x, y), ..., \phi^{(6)}(x, y)$

Step III: $while\ \ x \le b,\ compute$

(i) $y'(x_0) = \phi(x_0, y_0)$;

(ii) $y''(x_0) = \phi'(x_0, y_0), y'''(x_0) = \phi''(x_0, y_0), ..., y^{(6)}(x_0) = \phi^{(6)}(x_0, y_0)$

(iii) $h = \left(\dfrac{720Tol}{y^{(6)}(x_0)}\right)^{1/6}$,

(IV) $x = x_0 + h$

(V) $if\ \ x > b, set\ \ x = b\ \ , and\ \ if\ \ (x - x_0) > h_{\max}\ \ , set\ \ x = x_0 + h_{\max}$

(VI) Compute y_{j+1} from Equation (9)

(VII) $Y(index + 1) = y_{j+1}, X(index + 1) = x\ \ , x_0 = x\ \ ; y_0 = y_{j+1}$,

(VIII) $index = index + 1$

Step IV: plot the solution

(i) Plot (X, verses Y)

The algorithm is easily adaptable on computer; we present it in MATLAB environment as given in Appendix A

V. NUMERICAL RESULTS

In this section, numerical results are presented for stiff and singularly perturbed test problems.

V.I. A stiff equation

Stiff differential equations involve rapidly increasing or decaying transient solution, which results in a difficulty for most of the numerical integrators. The algorithm was tested on the stiff problem taken from [6]

$$y'(x) = -100y(x) + 99e^{2x}, \qquad y(0) = 0 \tag{16}$$

which has exact solution

$$y(x) = \frac{33}{34}\left(e^{2x} - e^{-100x}\right),$$

The numerical and exact solutions are shown are shown in Fig.2 while the numerical solution error is shown in Fig.3. Moreover, the maximum absolute error E and the required number of grid points N for achieving specific tolerance Tol are presented in Table 1.

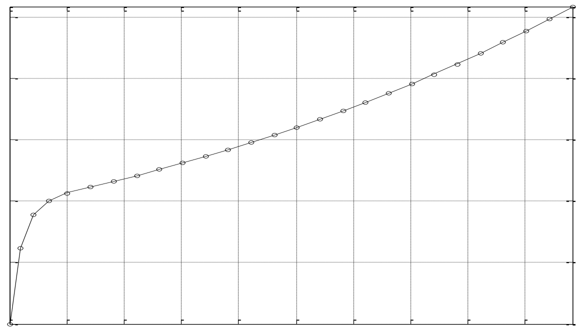

Fig.2. The exact solution (solid line) and the numerical one (circled) of problem (16) at $Tol = 10^{-3}$

Table 1. Maximum error E and number of grid points N for problem (16)

	$Tol = 10^{-3}$	$Tol = 10^{-4}$	$Tol = 10^{-5}$	$Tol = 10^{-6}$	$Tol = 10^{-7}$
E	1.8e-003	4.8e-004	9.5e-005	1.6e-005	2.6e-006
N	28	30	33	39	47

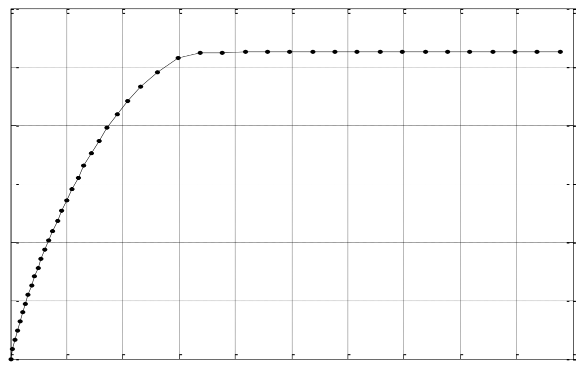

Fig.3. The numerical solution error of problem (16) at $Tol = 10^{-7}$

V.II. A stiff system

The above algorithm may be also applied to a system of equations. If we have $y, \phi(x, y) \in \mathbb{R}^m$ in (1), we have just to consider the formula in (9) for every component and take the minimum step size resulted from (15). Let be the stiff system taken from [4,11,12].

$$\begin{aligned} &y_1'(x) = 2y_1(x) + y_2(x) + 2\sin(t), \quad y_1(0) = 2, \\ &y_2'(x) = 998y_1(x) - 999y_2(x) + 999(\cos(t) - \sin(t)), \quad y_2(0) = 3, \end{aligned} \tag{17}$$

The exact solution is

$$y_1(x) = 2e^{-t} + \sin(t), y_2(x) = 2e^{-t} + \cos(t)$$

The numerical (dotted line) and exact (solid line) solutions are shown are shown in Fig.4 while the numerical solution error is shown in Fig.5. Moreover, the maximum absolute error E and the required number of grid points N for achieving the specific tolerance Tol are presented in Table 2. It is clear that the problem is solved using maximum allowed step size in the algorithm.

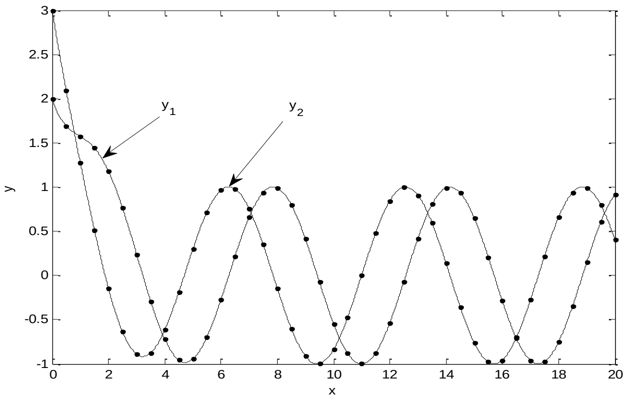

Fig.4. The exact solutions and the numerical solutions of problem (17) at $Tol = 10^{-3}$

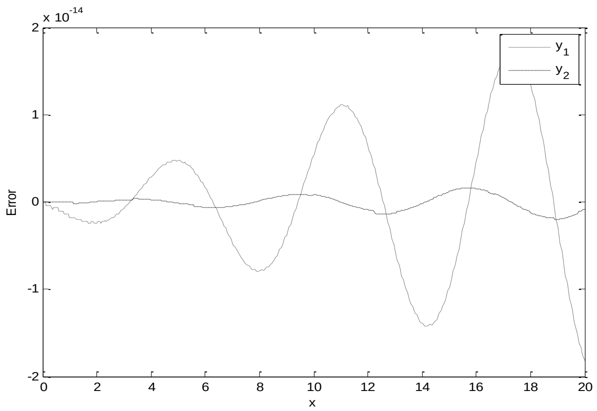

Fig.5. The numerical solution errors of problem (17) at $Tol = 10^{-7}$

Table 2. Maximum error E and number of grid points N for two components of problem (17)

E	$Tol = 10^{-3}$	$Tol = 10^{-4}$	$Tol = 10^{-5}$	$Tol = 10^{-6}$	$Tol = 10^{-7}$
$y_1(x)$	4.0e-014	4.0e-014	2.3e-014	4.9e-014	1.5e-014
$y_2(x)$	3.4e-016	3.4e-016	2.9e-016	1.6e-016	7.3e-016
N	501	502	505	603	804

V.III. singularly-perturbed problems

Now we consider the first nonlinear singularly perturbed IVP given by [8]

$$\varepsilon y'(x) = -\frac{1}{8} y(x)\left(y(x) - 20\right), \quad y(0) = 1, \tag{18}$$

The exact solution is

$$y(x) = \frac{20}{1 + 19e^{(-x/4\varepsilon)}}$$

This problem exhibits an initial layer near $x = 0$. Table 3 shows the maximum errors E and the required number of grid points N obtained with our algorithm when the integration is performed on the interval [0, 1] for a small value of the perturbation parameter $\varepsilon = 10^{-6}$.

Table 3. Error E and grid points N for problem (18) at $\varepsilon = 10^{-6}$

	$Tol = 10^{-3}$	$Tol = 10^{-4}$	$Tol = 10^{-5}$	$Tol = 10^{-6}$	$Tol = 10^{-7}$
E	4.9e-003	5.4e-004	5.9e-005	7.6e-006	9.6e-007
N	67	74	83	96	117

The second nonlinear singularly perturbed IVP given by [13]

$$\varepsilon y'(x) + y - x^2 y^2 = 0 \ , \quad y(0) = 1, \tag{19}$$

The exact solution is

$$\frac{e^{-x/\varepsilon}}{(x^2 + 2\varepsilon x + 2\varepsilon^2)e^{-x/\varepsilon} + (1 - 2\varepsilon^2)}$$

Table 5 shows the numerical results obtained over the interval [0, 1] for $\varepsilon = 10^{-6}$. In Fig. 6 the errors are shown when $\varepsilon = 10^{-6}$ .We note that even near $x = 0$, the algorithm performs very well, even with the presence of the initial layer.

VI. CONCLUSIONS

In this article we have developed an adaptive step-size nonlinear explicit integration algorithm for solving ordinary differential equations. The algorithm is based on a new fifth order nonlinear explicit integration scheme. The scheme has larger stability region than classical Rung-Kutta method. Several test problems are solved and the numerical results are presented in tables and figures. The results confirm that the algorithm is effective and appropriate for solving stiff and singularly perturbed ordinary differential equations.

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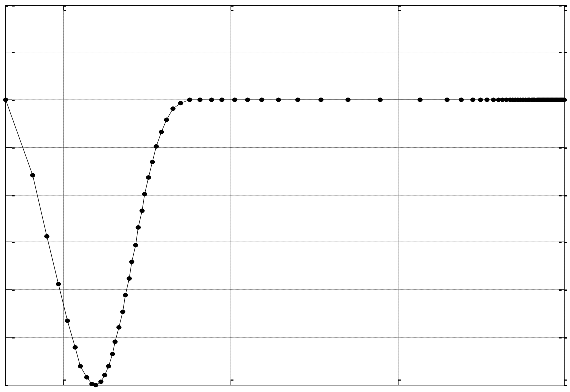

Fig.6. The numerical solution error for problem (19), at $\varepsilon = 10^{-6}$, $Tol = 10^{-7}$.

Table 5. Error E and grid points N for problem (19) at $\varepsilon = 10^{-6}$

	$Tol = 10^{-3}$	$Tol = 10^{-4}$	$Tol = 10^{-5}$	$Tol = 10^{-6}$	$h_{\max} = .02$
E	4.0e-004	7.1e-005	1.1e-005	1.9e-006	2.9e-007
N	71	74	79	85	95

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