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# Cubic Spline Solution of Nonlinear Singularly Perturbed Boundary Value Problems via Initial Value Method

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**Abstract**

In this paper, a cubic spline solution of nonlinear singularly perturbed two-point boundary value problems exhibiting boundary layers is presented via initial value method. The method is distinguished by the following fact: The original problem is approximated asymptomatically by two first order unperturbed initial value problems which are, in turn, solved using polynomial cubic spline approximation method. Error estimate and numerical results are presented to assess the accuracy and the performance of the method. It is observed that the present method approximates the exact solution very well overall the problem entire domain.

**Keywords**: Two point boundary layer problems; Initial value methods; Cubic spline.

## I. INTRODUCTION

Singularly perturbed boundary value problems (SPBVPs) arise frequently in various areas of applied science and engineering such as fluid mechanics, fluid dynamics, heat transfer, optimal control, chemical reactions, elasticity, etc. These problems are stiff and exhibit solutions with boundary or interior layers. Due to the stiffness and the layers' behavior, applying any standard numerical method on these problems yields poor solution with unacceptable oscillations. Moreover, as the perturbation parameter tends to zero the stiffness ratio increases and consequently more and significant computational difficulties are arising. Thus more efficient and adaptable computational techniques are required to solve SPBVPs. A large number of fitted mesh methods, fitted operator methods and special-purpose methods reflected the qualitative behavior of the solution layers have been recently proposed to provide accurate numerical solutions for SPBVPs (cf. [1–34]). However, most of these methods are developed for linear SPBVPs and for nonlinear ones quasilinearization technique is used. One of the most recent effective and attractive methods for solving SPBVPs is spline approximation method [19-34]. Spline approximation possesses attractive properties: piecewise smooth, compact support, differentiability, linear combination, which leads to linear algebraic systems that are easier to be solved. In this paper, we are interested in solving nonlinear two-point SPBVPs using cubic spline approximation via initial value method. The method is distinguished by the following fact: The original problem is approximated asymptomatically by two first order unperturbed initial value problems which are, in turn, solved using polynomial cubic spline method. Error estimate and numerical results are presented to assess the accuracy and the performance of the method. It is observed that the present method approximates the exact solution very well.

## II. DESCRIPTION OF THE METHOD

Consider the nonlinear SPBVP

$$\varepsilon\frac{d^2y}{dx^2}+p(x,y)\frac{dy}{dx}-q(x,y)=h(x)\,,\qquad x\in[0,1],\tag{1}$$

with boundary conditions

$$y(0)=\alpha,\quad y(1)=\beta\,,\tag{2}$$

where $0<\varepsilon\ll 1$ , $\alpha$ and $\beta$ are given constants, $p(x,y)$ , $q(x,y)$ and $h(x)$ are assumed to be sufficiently continuously differentiable functions, and $p(x,y)\geq M>0$ for $x\in[0,1]$ where $M$ is some positive constant. Under these assumptions, SPBVP (1)-(2) has a solution which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x=0$ . The SPBVP (1)-(2) is approximated asymptomatically by two first order unperturbed initial value problems (IVPs) using the following theorem.

Theorem 1. [14]. The solution $y(x)$ of the nonlinear SPBVP (1)-(2) can be approximated asymptotically by:

$$y(x)=u(x)+v(t)+O(\varepsilon),\qquad t=\frac{x}{\varepsilon}\,,\tag{3}$$

where $u(x)$ and $v(t)$ are the solutions of the outer and inner region problems given, respectively, by:

$$\begin{cases} p(x,u)\ \dfrac{du}{dx} - q(x,u) = h(x), \\ u(1) = \beta, \end{cases} \tag{4}$$

and

$$\begin{cases} \dfrac{dv}{dt} + f(0, u(0)+v(t)) = f(0, u(0)) \\ v(0) = \alpha - u(0) \end{cases}, \tag{5}$$

where

$$f(0, u(0)+v(t)) = \int p(0, u(0)+v(t))\, dv$$

Thus, in order to obtain a smooth approximate solution of the SPBVP (1)-(2), we only need to obtain smooth approximate solutions for the two first-order unperturbed IVPs (4)-(5).

### II.I. Polynomial Cubic Spline method for IVPs

In this subsection we describe polynomial cubic spline approximation method for solving IVPs in the general form:

$$\frac{d\theta}{dr} = \phi(r,\theta), \quad \theta(r_0) = \theta_0. \tag{6}$$

Let us consider a uniform mesh nodal points $r_i$ on $[r_0, r_N]$ such that $r_i = ih, i = 0,1,2,..,N$ , $h = \dfrac{r_N - r_0}{N}$ and $N$ is the number of subintervals. The cubic spline interpolate function $S(r)$ in $r_i \le r \le r_{i+1}$ in terms of its first derivatives $S'(r_i) = m_i$ is given by [20,23,25,27]

$$\begin{aligned} S(r) &= \frac{m_i(r_{i+1}-r)^2(r-r_i)}{h^2} - \frac{m_{i+1}(r-r_i)^2(r_{i+1}-r)^2}{h^2} \\ &+ \frac{\theta_i(r_{i+1}-r)^2[2(r-r_i)+h]}{h^3} + \frac{\theta_{i+1}(r-r_i)^2[2(r_{i+1}-r)+h]}{h^3}. \end{aligned} \tag{7}$$

Differentiating (7) with respect to $r$ and simplifying, we have

$$\begin{aligned} S'(r) &= \frac{m_i(r_{i+1}-r)(2r_i + r_{i+1} - 3r)}{h^2} \\ &- \frac{m_{i+1}(r-r_i)(r_i + 2r_{i+1} - 3r)}{h^2} + \frac{6(\theta_{i+1}-\theta_i)(r-r_i)(r_{i+1}-r)}{h^3}. \end{aligned} \tag{8}$$

Again, differentiating (8) with respect to $r$ , we have

$$\begin{aligned} S''(r) &= \frac{-2m_i(r_i + 2r_{i+1} - 3r)}{h^2} - \frac{2m_{i+1}(2r_i + r_{i+1} - 3r)}{h^2} \\ &+ \frac{6(\theta_{i+1}-\theta_i)(r_i + r_{i+1} - 2r)}{h^3} \end{aligned}, \tag{9}$$

which gives

$$S''(r_{i+1}) = \frac{2m_i}{h} + \frac{4m_{i+1}}{h} - \frac{6(S_{i+1}-S_i)}{h^2}. \tag{10}$$

Differentiating (6) with respect to $r$ and using (10) we get

$$\begin{aligned} &\frac{2\phi(r_i,S_i)}{h} + \frac{4\phi(r_{i+1},S_{i+1})}{h} - \frac{6(S_{i+1}-S_i)}{h^2} = \\ &\phi_r(r_{i+1},S_{i+1}) + \phi_\theta(r_{i+1},S_{i+1})\phi(r_{i+1},S_{i+1}),\ i = 0,1,2,..,N,\ S_0 = \theta_0 \end{aligned}, \tag{11}$$

from which $S_{i+1}$ can be computed to get the cubic spline solution $S(r)$ with bounded error given by the following theorem.

Theorem2. [20,23,25,27]. If $\theta(r) \in C^4[r_0, r_N]$ , and $h$ is a sequence of partitions on $[r_0, r_N]$ with $\lim_{N\to\infty} h = 0$ , then we have for the interpolate cubic spline $S(r)$ , uniformly for $r_0 \le r \le r_N$

$$|\theta(r) - S(r)| \le Ch^4 \tag{12}$$

Applying the described cubic spline approximation method on the two IVPs (4)-(5) results in a smooth approximate solution of the original SPBVP (1).

### II.II. Error analysis

The error of the present method has two sources: one from the asymptotic approximation and the other from the cubic interpolation. Let $h_o$ and $h_{in}$ be the mesh spacing on the non-boundary layer and the boundary layer domain respectively. Then we have the following bounded errors.

#### II.II.I. *Error on the outer region domain*

Let $y(x)$ be the exact solution of the original problem (1)-(2), $u(x)$ be the exact solution of the reduced problem (4), and $S_o(x)$ be the cubic spline solution of (4). Then on the non-boundary layer domain, the error is

$$\begin{aligned} |y(x) - S_o(x)| &\le |y(x) - u(x)| + |u(x) - S_o(x)| = \\ &= O(\varepsilon) + O(h_o^4) \end{aligned}. \tag{13}$$

In more times, the exact solution of the reduced problem can be easily obtained and the second term of the above error inequality is vanished.

#### II.II.II. *Error on the inner region domain*

Let $w(x)$ be the exact solution of boundary layer problem (5), and $S_{in}(x)$ be the cubic spline solution of (5). Then on the boundary layer domain, the error is

$$\begin{aligned} |y(x) - S_{in}(x)| &\le |y(x) - w(x)| + |w(x) - S_{in}(x)| = \\ &= O(\varepsilon) + O(h_{in}^4) \end{aligned}, \tag{14}$$

### III. NUMERICAL RESULTS

To demonstrate the applicability and the accuracy of the method, some SPBVPs are solved and numerical results are presented in tables and figures. These SPBVPs are widely discussed in the literature and their uniformly valid approximate solutions are available for comparison.

Example 1**.** Consider the SPBVP from Bender and Orszag [1] given by

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0, \tag{15}$$

with $y(0)=0, y(1)=0$. The problem (15) has a uniformly valid approximation [1] for comparison,

$$y(x)=\ln(2/(1+x))-(\ln 2)e^{-2x/\varepsilon}+O(\varepsilon).$$

The outer and inner region problems are given by

$$\begin{cases} 2u'(x)+e^{u(x)}=0, & u(1)=0 \\ \dfrac{dv}{dt}+2v(t)=0, & v(0)=-u(0). \end{cases} \quad (16)$$

The computational results are presented in Tables 1 and 2 at $h_o=h_{in}=0.2$ for $\varepsilon=10^{-2},10^{-3}$, respectively.

**Table 1**. Results for Example 1 at $h_o=h_{in}=0.2$, $\varepsilon=10^{-2}$.

| $x$ | Present method | Solution in[1] | Error |
|---|---|---|---|
| 0.0010 | 0.12464676709 | 0.12521453874 | 5.6777 e-04 |
| 0.0110 | 0.60540434370 | 0.60977249790 | 4.3682 e-03 |
| 0.0210 | 0.66197049958 | 0.66680407158 | 4.8336 e-03 |
| 0.0300 | 0.66673285673 | 0.66187023823 | 4.8626 e-03 |
| 0.1000 | 0.59783700075 | 0.59783699932 | 0.0000 e-11 |
| 0.2000 | 0.51082562376 | 0.51082562376 | 0.0000 e-14 |
| 0.3000 | 0.43078291609 | 0.43078291609 | 0.0000 e-16 |
| 0.4000 | 0.35667494393 | 0.35667494393 | 0.0000 e-16 |
| 0.5000 | 0.28768207245 | 0.28768207245 | 0.0000 e-16 |
| 0.6000 | 0.22314355131 | 0.22314355131 | 0.0000 e-16 |
| 0.7000 | 0.16251892949 | 0.16251892949 | 0.0000 e-16 |
| 0.8000 | 0.10536051565 | 0.10536051565 | 0.0000 e-16 |
| 0.9000 | 0.05129329438 | 0.05129329438 | 0.0000 e-16 |
| 1.0000 | 0.00000000000 | 0.00000000000 | 0.0000 e-16 |

**Table 2**. Results for Example 1, $h_o=h_{in}=0.2$, $\varepsilon=10^{-3}$

| $x$ | Present method | Solution in[1] | Error |
|---|---|---|---|
| 0.0001 | 0.12554627242 | 0.12530098562 | 2.4529 e-04 |
| 0.0011 | 0.61524488830 | 0.61564411132 | 3.9922 e-04 |
| 0.0021 | 0.68065524068 | 0.68113457158 | 4.7933 e-04 |
| 0.0031 | 0.68864528152 | 0.68914045395 | 4.9517 e-04 |
| 0.0040 | 0.68942016707 | 0.68892263431 | 4.5793 e-04 |
| 0.1000 | 0.59783700075 | 0.59783700075 | 0.0000 e-16 |
| 0.2000 | 0.51082562376 | 0.51082562376 | 0.0000 e-16 |
| 0.3000 | 0.43078291609 | 0.43078291609 | 0.0000 e-16 |
| 0.4000 | 0.35667494393 | 0.35667494393 | 0.0000 e-16 |
| 0.5000 | 0.28768207245 | 0.28768207245 | 0.0000 e-16 |
| 0.6000 | 0.22314355131 | 0.22314355131 | 0.0000 e-16 |
| 0.7000 | 0.16251892949 | 0.16251892949 | 0.0000 e-16 |
| 0.8000 | 0.10536051565 | 0.10536051565 | 0.0000 e-16 |
| 0.9000 | 0.05129329438 | 0.05129329438 | 0.0000 e-16 |
| 1.0000 | 0.00000000000 | 0.00000000000 | 0.0000 e-16 |

Example 2**.** Consider the nonlinear SPP from O'Malley [2] given by

$$\varepsilon y''(x)+e^{y(x)}y'(x)-\frac{\pi}{2}\sin(\frac{\pi x}{2})e^{2y(x)}=0\ , \quad (17)$$

with $y(0)=0$ and $y(1)=0$ .The problem has a uniformly valid approximation for comparison[2]

$$y(x)=-\ln\left[(1+\cos(\pi x/2))(1-0.5e^{-x/2\varepsilon})\right]+O(\varepsilon).$$

The outer and inner region problems are given by

$$\begin{cases} e^{u(x)}u'(x)-\dfrac{\pi}{2}\sin(\pi x/2)e^{2u(x)}=0, & u(1)=0 \\ \dfrac{dv}{dt}+e^{u(0)+v(t)}=e^{u(0)}, & v(0)=-u(0). \end{cases} \quad (18)$$

The computational results are presented in Tables 3 and 4 at $h_o=h_{in}=0.2$ for $\varepsilon=10^{-2},10^{-3}$, respectively.

**Table 3**. Results for Example 2 at $h_o=h_{in}=0.2$, $\varepsilon=10^{-2}$.

| $x$ | Present method | Solution in[2] | Error |
|---|---|---|---|
| 0.0001 | -0.04761798081 | -0.0475799500 | 3.8031e-05 |
| 0.0011 | -0.35272794896 | -0.35279134351 | 6.3395e-05 |
| 0.0021 | -0.50054097170 | -0.50074234655 | 2.0137e-04 |
| 0.0031 | -0.58036612761 | -0.58075262680 | 3.8650e-04 |
| 0.0041 | -0.62557761970 | -0.62617649880 | 5.9888e-04 |
| 0.0051 | -0.65171896274 | -0.65254502419 | 8.2606e-04 |
| 0.0060 | -0.66675410455 | -0.66571707895 | 1.0370e-03 |
| 0.1000 | -0.68697232562 | -0.68359766435 | 3.3747e-03 |
| 0.2000 | -0.66837102908 | -0.66834832885 | 2.2700e-05 |
| 0.3000 | -0.63710923979 | -0.63710908684 | 1.5295e-07 |
| 0.4000 | -0.59278360071 | -0.59278359968 | 1.0306e-09 |
| 0.5000 | -0.53479999673 | -0.53479999673 | 6.9440e-12 |
| 0.6000 | -0.46234012212 | -0.46234012212 | 4.6685e-14 |
| 0.7000 | -0.37431184521 | -0.37431184521 | 3.3307e-16 |
| 0.8000 | -0.26927646955 | -0.26927646955 | 0.0000e-16 |
| 0.9000 | -0.14534153445 | -0.14534153445 | 0.0000e-16 |
| 1.0000 | 0.00000000000 | 0.00000000000 | 0.0000e-16 |

**Table 4**. Results for Example 2 at $h_o=h_{in}=0.2$, $\varepsilon=10^{-3}$

| $x$ | Present method | Solution in[2] | Error |
|---|---|---|---|
| 0.0001 | -0.047618591 | -0.047579958 | 3.8633e-05 |
| 0.0011 | -0.35280184 | -0.352802536 | 6.9421e-07 |
| 0.0021 | -0.50081029 | -0.500811965 | 1.6703e-06 |
| 0.0031 | -0.58095305 | -0.580956614 | 3.5634e-06 |
| 0.0041 | -0.62660435 | -0.626610122 | 5.7675e-06 |
| 0.0051 | -0.65330777 | -0.653315882 | 8.1074e-06 |
| 0.0061 | -0.66915990 | -0.669170405 | 1.0505e-05 |
| 0.0071 | -0.67864962 | -0.678662552 | 1.2923e-05 |
| 0.0081 | -0.68435735 | -0.684372704 | 1.5348e-05 |
| 0.0090 | -0.68754476 | -0.687527233 | 1.7529e-05 |
| 0.1000 | -0.68697232 | -0.686972325 | 0.0000e-11 |
| 0.2000 | -0.66837102 | -0.668371029 | 0.0000e-13 |
| 0.3000 | -0.63710923 | -0.637109239 | 0.0000e-15 |
| 0.4000 | -0.59278360 | -0.592783600 | 0.0000e-16 |
| 0.5000 | -0.53479999 | -0.534799996 | 0.0000e-16 |
| 0.6000 | -0.462340122 | -0.462340122 | 0.0000e-16 |
| 0.7000 | -0.374311845 | -0.374311845 | 0.0000e-16 |
| 0.8000 | -0.269276469 | -0.269276469 | 0.0000e-16 |
| 0.9000 | -0.145341534 | -0.145341534 | 0.0000e-16 |
| 1.0000 | 0.000000000 | 0.000000000 | 0.0000e-16 |

Example 3. Consider the nonlinear SPP from Johnson [3]

$$\varepsilon y''(x)+(x+y)\big(y'(x)-x-\varepsilon(x+y)\big)=\varepsilon\ , \quad (19)$$

with boundary conditions $y(0)=0 \; and \; y(1)=5/2$. The solution is approximated in [3] as

$$y(x)=\frac{x^2}{2}+2\tanh(x/\varepsilon)+O(\varepsilon).$$

The outer and inner region problems are given by

$$\begin{cases}(x+u)(u'(x)-x)=0, \quad u(1)=5/2\\ \dfrac{dv}{dt}+x(u(0)+v(t))+0.5(u(0)+v(t))^2=0.5(u(0))^2, \quad (20)\\ v(0)=-u(0).\end{cases}$$

The computational results are presented in Tables 5 and 6 at $h_o=h_{in}=0.2$ for $\varepsilon=10^{-2},10^{-3}$, respectively.

**Table 5**. Results for Example 3 at $h_o=h_{in}=0.2$, $\varepsilon=10^{-2}$ .

| $x$ | Present method | Solution in[3] | Error |
|---|---|---|---|
| 0.0001 | 0.19933648924 | 0.19967108441 | 3.3460 e-04 |
| 0.0011 | 1.60105854352 | 1.60371647484 | 2.6579 e-03 |
| 0.0021 | 1.94112437322 | 1.94241671262 | 1.2923 e-03 |
| 0.0030 | 1.99088442765 | 1.99055950737 | 3.2492 e-04 |
| 0.1000 | 2.00500000000 | 2.00499999175 | 8.2446e-09 |
| 0.2000 | 2.02000000000 | 2.02000000000 | 0.0000 e-14 |
| 0.3000 | 2.04500000000 | 2.04500000000 | 0.0000 e-16 |
| 0.4000 | 2.08000000000 | 2.08000000000 | 0.0000 e-16 |
| 0.5000 | 2.12500000000 | 2.12500000000 | 0.0000 e-16 |
| 0.6000 | 2.18000000000 | 2.18000000000 | 0.0000 e-16 |
| 0.7000 | 2.24500000000 | 2.24500000000 | 0.0000 e-16 |
| 0.8000 | 2.32000000000 | 2.32000000000 | 0.0000 e-16 |
| 0.9000 | 2.40500000000 | 2.40500000000 | 0.0000 e-16 |
| 1.0000 | 2.50000000000 | 2.50000000000 | 0.0000 e-16 |

**Table 6**. Results for Example 3 at $h_o=h_{in}=0.2$, $\varepsilon=10^{-3}$.

| $x$ | Present method | Solution in[3] | Error |
|---|---|---|---|
| 0.0001 | 0.19933599424 | 0.19958730764 | 2.5131 e-04 |
| 0.0011 | 1.60099864852 | 1.60121899862 | 2.2035 e-04 |
| 0.0021 | 1.94090607822 | 1.94101414802 | 1.0807 e-04 |
| 0.0031 | 1.99190352344 | 1.99193384573 | 3.0322 e-04 |
| 0.0040 | 1.99867356698 | 1.99866659947 | 1.9675 e-05 |
| 0.1000 | 2.00500000000 | 2.00500000000 | 0.0000 e-15 |
| 0.2000 | 2.02000000000 | 2.02000000000 | 0.0000 e-16 |
| 0.3000 | 2.04500000000 | 2.04500000000 | 0.0000 e-16 |
| 0.4000 | 2.08000000000 | 2.08000000000 | 0.0000 e-16 |
| 0.5000 | 2.12500000000 | 2.12500000000 | 0.0000 e-16 |
| 0.6000 | 2.18000000000 | 2.18000000000 | 0.0000 e-16 |
| 0.7000 | 2.24500000000 | 2.24500000000 | 0.0000 e-16 |
| 0.8000 | 2.32000000000 | 2.32000000000 | 0.0000 e-16 |
| 0.9000 | 2.40500000000 | 2.40500000000 | 0.0000 e-16 |
| 1.0000 | 2.50000000000 | 2.50000000000 | 0.0000 e-16 |

By considering the given problems solutions as our exact solution, the numerical solution errors are provided for all examples through tables 1-6. The results indicate that the proposed method approximates the solution very well overall the entire domain.

## IV. CONCLUSIONS

We have presented a cubic spline solution of nonlinear singularly perturbed two-point boundary value problems exhibiting boundary layers via initial value method. The original problem is approximated asymptomatically by two first order unperturbed initial value problems namely outer and inner region problems. The solution of the original problem is a combination of the outer and inner region solutions. A polynomial cubic spline approximate method is employed to solve these IVPs to get a smooth approximate solution of the original problem. The sources of the error and the error estimate are presented. The method has the ability in solving the considered problem with no need to linearization techniques to convert the problem into a sequence of linear ones to be solved iteratively. We have implemented the method on three nonlinear examples taking different values of $\varepsilon$ and have tabulated the results at non-nodal grid points to be compared with solutions presented in literature. The numerical results indicate that the present method can handles the considered problem effectively and approximates the exact solution very well overall the entire domain.

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