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Analytical solutions of Fixed-end Nano-beam with Surface Elasticity Subjected to Uniform Shear Load

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Abstract

In this study, the effects of surface elasticity on both ends fixed nano-beams subjected to uniform shear load are studied in the frame of surface elasticity theory. A set of analytical solutions is obtained by Ariy stress function method. As a example, the deformations induced, by an uniformly shear loaded isotropic nano-beam with rectangular cross section under classical boundary conditions. The results indicate some interesting characteristics, which are distinctly different from those in classical elasticity beam theory.

Keywords: Surface elasticity, Fixed-end Nano-beam, Shear load, Ariy stress function method, Analytical solutions

1. INTRODUCTION

At nano-scale, because of the increasing ration of surface area to volume, surface effects often play an important role in the mechanical performance of nano-beam. To account for the influence of surface effects in solid mechanics, Gurtin and Murdoch [1,2] developed the theory of surface elasticity by using the continuum mechanics. For some elementary deformation modes, the prediction of surface elasticity showed a good agreement with directly atomic simulation [3,4]. Therefore, the surface elasticity theory has been widely adopted to investigate the mechanical phenomena at nano-scale. Both theoretical and experimental studies have been undertaken to investigate surface stress effects in micro-beams[5-7]. Lagowski et al. analyzed the influence of residual surface stress on the vibration of thin crystals [8]. Wang and Feng studied surface effects on the axial buckling and the transverse vibration of nanowires are examined by using the refined Timoshenko beam theory [9]. He and Lilley considered the static deformation of nanosize beams and investigated the influence of boundary conditions on the natural frequency nanosize beams [10,11]. Gurtin et al. thought that the residual surface stress would induce a distributed traction over the upper and lower surfaces of beam under bending, in addition to the compressive axial force [12]. Park analyzed the size-dependent effect of the residual surface stress on the resonant frequencies of nanowires under finite deformation [13]. Zhang et al. considered the surface stress that is can be seen as an external loading and represented by a corresponding equivalent uniformly distributed loading along the beam span [14].

In this paper, using Ariy stress function method, we deduce the effects of elasticity on the both ends fixed nano-beams subjected to uniform shear load. This method allows us to easily extend our analysis to problems involving fixed-end nano-beam subjected to shear load on a finite region.

2. Basic equations of surface elasticity

In surface elasticity theory, the equilibrium and constitutive equations in the bulk of material are the same as those in classical elastic theory. In the absence of body force, the equilibrium equations, constitutive law and geometry relations in the bulk are as follows.

$$\sigma_{ij,j} = 0 \tag{1}$$

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{v}{1-2v}\varepsilon_{kk}\delta_{ij}\right) \tag{2}$$

where G and v are the shear modulus and Poisson's ratio of the bulk material, σ_{ij} and ε_{ij} are the stress tensor and strain tensor in the bulk material, respectively.

The strain tensor is related to the displacement vector u_i by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

Assume that the surface of the material adheres perfectly to its bulk without slipping. Then the equilibrium conditions on the surface are expressed as

$$\sigma_{\beta\alpha}n_\beta + \sigma^s_{\beta\alpha,\beta} = 0 \tag{4}$$

$$\sigma_{ij}n_in_j = \sigma^s_{\alpha\beta}\kappa_{\alpha\beta} \tag{5}$$

where n_i denotes the normal to the surface, $\kappa_{\alpha\beta}$ the curvature tensor of the surface, and $\sigma^s_{\alpha\beta}$ the surface stress tensor.

The surface stress tensor is related to the surface strain tensor by

$$\sigma^s_{\alpha\beta} = \tau^s \delta_{\beta\alpha} + 2\left(\mu^s - \tau^s\right)\varepsilon_{\beta\alpha} + \left(\lambda^s + \tau^s\right)\varepsilon_{\gamma\gamma}\delta_{\beta\alpha} \qquad (6)$$

where τ^s is the residual surface tension under unstrained conditions, μ^s and λ^s are surface Lam´e constants which can be determined by atom simulations or experiments [15].

3. BASIC EQUATIONS IN PLAN STRESS STATE

As in classical theory of elasticity, we define the Airy stress function $\phi(x,z)$ by

$$\sigma_{11} = \frac{\partial^2\phi}{\partial z^2}, \sigma_{33} = \frac{\partial^2\phi}{\partial x^2}, \sigma_{13} = -\frac{\partial^2\phi}{\partial x\partial z} \qquad (7)$$

For the considered plane problem, the equilibrium equations and Hooke's law in the bulk reduce to

$$\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{13}}{\partial z} = 0, \frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{33}}{\partial z} = 0 \qquad (8)$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \upsilon\sigma_{33}\right]$$
$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \upsilon\sigma_{11}\right] \qquad (9)$$
$$\varepsilon_{13} = \frac{2(1+\upsilon)}{E}\sigma_{13}$$

where E and υ are Young's modulus and Poisson ratio, respectively. The strains are related to the displacements by

$$\varepsilon_{11} = \frac{\partial u}{\partial x},\ \varepsilon_{33} = \frac{\partial w}{\partial z},\ \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad (10)$$

which satisfy the following compatibility condition

$$\frac{\partial^2\varepsilon_{11}}{\partial z^2} + \frac{\partial^2\varepsilon_{33}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{13}}{\partial x\partial z} \qquad (11)$$

Then the equilibrium equations in Eq.（11）are satisfied automatically and the compatibility equation in Eq.（8）becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0 \qquad (12)$$

4. FIXED-END NANO-BEAMS SUBJECTED TO UNIFORM SHEAR LOAD

Now we use the above surface elasticity theory to examine the influence of surface elasticity on a fixed-end nano-beam with unit width rectangular cross section subjected to shear load $q(x)$. The length of the beam is l and height *h*. We refer to a Cartesian coordinate system (*o- xyz*), as shown in Fig. 1, where the *x* axis is along the surface, and the *z* axis perpendicular to the surface. Take the stress function in the following form of a bi-harmonic polynomial with 7 terms

$$\phi(x,z) = a\left(\frac{1}{5}z^5 - x^2z^3\right) + bxz^3 + cz^3 + dz^2 + ex^2z + fxz + gx^2 \tag{13}$$

where *a,b,c,d,e,f* and *g* are unknown constants to be determined. The stress and displacement components are given

$$\begin{aligned}
\sigma_{xx} &= 2a\left(2z^3 - 3x^2z\right) + 6bxz + 6cz + 2d \\
\sigma_{xz} &= 6axz^2 - 3bz^2 - 2ex - f \\
\sigma_{zz} &= -2az^3 + 2ez + 2g
\end{aligned} \tag{14}$$

$$\begin{aligned}
u(x,z) &= \frac{1}{E}\left[2a(2+\upsilon)xz^3 - \left(2ax^3 - 3bx^2 - 6cx + 2\upsilon ex\right)z + 2dx - 2\upsilon gx\right] \\
&\quad - \frac{1}{E}\left[b(2+\upsilon)z^3 + 2(1+\upsilon)fz\right] + \omega z + u_0 \\
w(x,z) &= \frac{1}{E}\left[-2a(1+2\upsilon)z^4 + \left(3a\upsilon x^2 - 3b\upsilon x - 3\upsilon c + e\right)z^2 - 2d\upsilon z + 2gz\right] \\
&\quad + \frac{1}{E}\left[\frac{1}{2}ax^4 - bx^3 - 3cx^2 - (2+\upsilon)ex^2\right] - \omega x + w_0
\end{aligned} \tag{15}$$

where ω is an arbitrary constants, u_0 and w_0 denote the translation and rotation of rigid body, respectively.

In the case where there is a uniform shear load q_0 (Fig.1) acting over the region $|x| \le l$. The plane-strain conditions are assumed with $\varepsilon_{2i} = 0$, and the contact is assumed to be frictionless. In this case, the stress boundary conditions Eqs. (4) and (5)

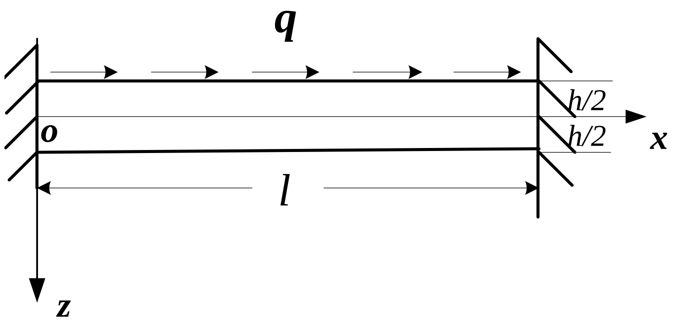

Fig.1 Fixed-end beam subjected to uniform shear load

on the contact surface *(z = -h/2)* are simplified to

$$\sigma_{zz} = 0 \tag{16}$$

$$\sigma_{xz} + q(x) = -\left(\frac{d\tau^s}{dx} + k^s \frac{d^2u}{dx^2} \right) \tag{17}$$

where $q(x)$ is the external shear load applied on the surface, and $k^s = (2\mu^s + \lambda^s)$ is a surface material constant.

Timoshenko and Goodier [16] presented the methods for dealing with the boundary conditions for fixed-end beams. The method is to treat the displacement boundary conditions as, (1) $z = \pm \frac{h}{2}$, $\sigma_{zz} = 0$,(2) $z = h/2$, $\sigma_{xz} = -q_0 - k^s \frac{d^2u}{dx^2}$,(3) $z = -h/2$, $\sigma_{xz} = 0$,(4) $x = 0, z = 0$ point and $x = l, z = 0$ point, $u=w=0$; $\partial u/\partial z = 0$; $\partial w/\partial x = 0$.

By substituting the stress components Eqs. (14) and displacement components Eqs. (15) into corresponding boundary conditions, 10 algebraic equations can be obtained and all the unknown constants can be determined as

$$\begin{gathered}
a = -\frac{4q_0(v+1)\exp(1)}{\left[5(v+1)\left(l + \frac{4}{5}x - \frac{6}{5}\right)h^2 + 4l^3 - 4l^2\right]\exp(1) + 12hk^s(v+1)(l+2x-2)} \\
b = \frac{4q_0(v+1)(l-2)\exp(1)}{\left[5(v+1)\left(l + \frac{4}{5}x - \frac{6}{5}\right)h^2 + 4l^3 - 4l^2\right]\exp(1) + 12hk^s(v+1)(l+2x-2)} \\
c = \frac{q_0(v+1)\left[(v+2)h^2 - 6l^2 + 8l\right]\exp(1)}{\left[5(v+1)\left(l + \frac{4}{5}x - \frac{6}{5}\right)h^2 + 12l^3 - 12l^2\right]\exp(1) + 36hk^s(v+1)(l+2x-2)} \\
e = 0 \\
f = \frac{2q_0\left[(v+1)h^2 + 2l^2 - 2l\right]l\exp(1)}{\left[5(v+1)\left(l + \frac{4}{5}x - \frac{6}{5}\right)h^2 + 4l^3 - 4l^2\right]\exp(1) + 12hk^s(v+1)(l+2x-2)} \\
g = 0
\end{gathered} \tag{18}$$

Then, stress and displacement components are then obtained

$$
\begin{aligned}
\sigma_{xx} &= \frac{2(v+1)\left[(v+2)h^2+2(6x+4-3l)l+12x(x-2)-8z^2\right]q_0\exp(1)z}{\left[(v+1)(5l+4x-6)h^2+4l^2(l-1)\right]\exp(1)+12hk^s(v+1)(l+2x-2)} \\
\sigma_{xz} &= -\frac{2\left[(v+1)(l-x)h^2+2l^2(l-1)+6z^2(v+1)\left(l+2(x-1)\right)\right]q_0\exp(1)}{\left[(v+1)(5l+4x-6)h^2+4l^2(l-1)\right]\exp(1)+12hk^s(v+1)(l+2x-2)} \\
\sigma_{zz} &= -\frac{2(v+1)\left(h^2-4z^2\right)q_0\exp(1)z}{\left[(v+1)(5l+4x-6)h^2+4l^2(l-1)\right]\exp(1)+12hk^s(v+1)(l+2x-2)} \\
u &= -\frac{1}{2G}\left\{\frac{\left\{2l^3+(3x-2)l^2+\left[\left(h^2+z^2\right)v+h^2-3x^2+2z^2-4x\right]l\right.}{\left[(l-1)l^2+(v+1)\left(x+\frac{5}{4}l-\frac{3}{2}\right)h^2\right]\exp(1)+3hk^s(v+1)(2x+l-2)}\right. \\
&\quad\left.+\frac{\left.\left[\left(2z^2-h^2\right)x-2z^2\right]v-2x^3+6x^2+\left(4z^2-h^2\right)x-4z^2\right\}q_0\exp(1)z}{\left[(l-1)l^2+(v+1)\left(x+\frac{5}{4}l-\frac{3}{2}\right)h^2\right]\exp(1)+3hk^s(v+1)(2x+l-2)}\right\} \\
w &= -\frac{1}{2G}\left\{\frac{\left\{-6\left(z^2v+x^2\right)l^2+\left[12\left(x+\frac{2}{3}\right)z^2v+4x^3+8x^2\right]l\right.}{\left[4(l-1)l^2+(v+1)(4x+5l-6)h^2\right]\exp(1)+12hk^s(v+1)(2x+l-2)}\right. \\
&\quad\left.+\frac{\left.h^2v^2z^2+2z^2\left(h^2+4x^2-8z^2-12x\right)v+h^2z^2+2x^4-8z^4-8x^3\right\}q_0\exp(1)z}{\left[(l-1)l^2+(v+1)\left(x+\frac{5}{4}l-\frac{3}{2}\right)h^2\right]\exp(1)+3hk^s(v+1)(2x+l-2)}\right\}
\end{aligned}
\tag{19}
$$

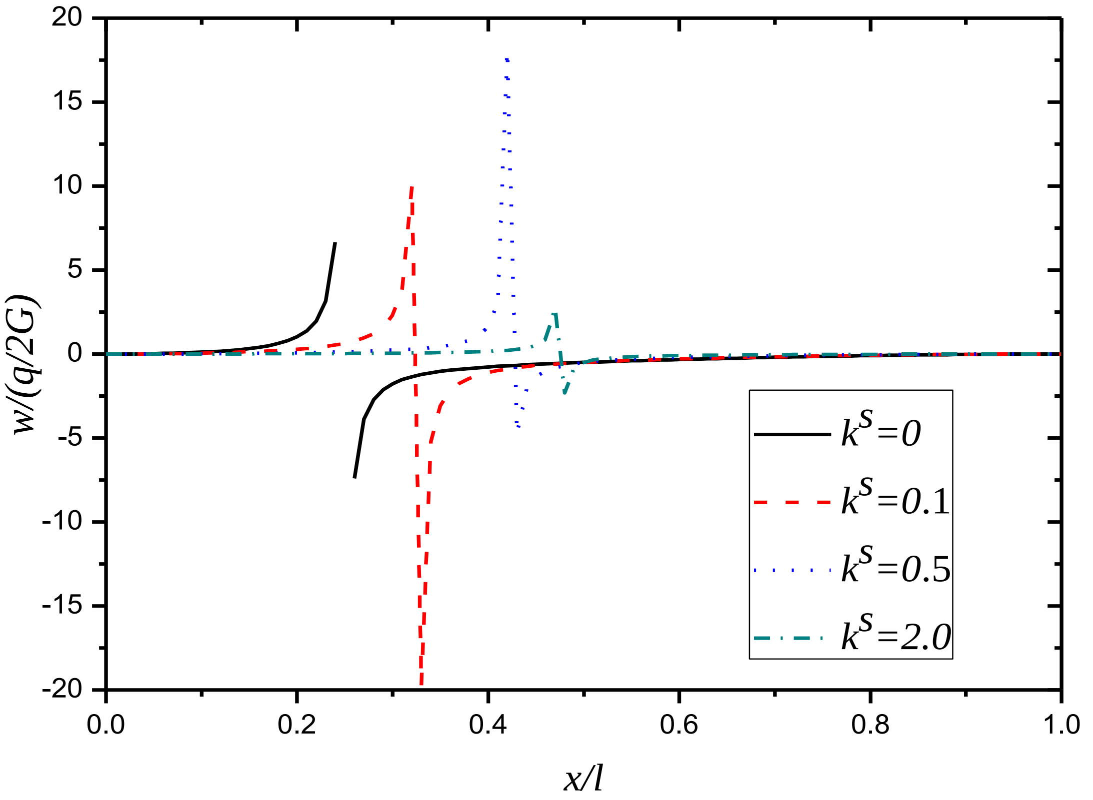

Figure 2: Distribution of the displacement w under uniform loading.

As show in Figure.2, the results indicated that the normal displacement w is a continuously distribution when the surface elasticity is considered by at the loading region, which is opposed to a singularity predicted by classical elasticity theory. In addition, the position of the maximum normal displacement in the bulk increases with an increase in surface elasticity. It is also found in Figure. 2 that the specific location as, $k^s = 0, x = 0.25$, $k^s = 0.1, x = 0.33$, $k^s = 0.5, x = 0.42$, $k^s = 2, x = 0.48$.

5. CONCLUSIONS

In this paper, we consider the influence of surface elasticity on the mechanical behavior of the fixed-end nano-beam subjected to uniform shear load. Through the Ariy stress function methodology, the general analytical solutions of nano-beam are derived rigorously. It is found out that surface elasticity theory illuminates some interesting characteristics of fixed-end beam on nanoscales, which are distinctly different form the classical solutions of elasticity without surface elasticity. Our results display that for nano-beam problems, the classical elasticity theory predicts some unreasonable results and therefore the effects of surface elasticity should be accounted.

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