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Some Theorems on T – Fuzzy Ideals of a $\ell-$Near-Ring

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Abstract

In this paper, we made an attempt to study the properties of family of $T-$fuzzy ideal of $\ell-$near-ring and we introduce some theorems in smallest $T-$fuzzy ideal of $\ell-$ near-ring in R.

Keywords: Fuzzy subset, $T-$fuzzy ideal, family of $T-$fuzzy ideal, smallest of $T-$fuzzy ideal and direct product of $T-$fuzzy ideal.

INTRODUCTION

The concept of fuzzy sets was initiated by L.A.Zadeh in 1965. After the introduction of fuzzy sets several researchers explored on the generalization of the concept of fuzzy sets. In 1971, W.J. Liu studied fuzzy ideals in rings and Bh. Satyanarayana introduced $\Gamma-$near-rings. In W.A. Dudek and Y.B. Jun introduced fuzzy subgroups over a $t-$norm. In M. Shabir and M. Hussan characterized the sum of fuzzy ideals. In Srinivas, Nagaiah and Narasimha Swamy studied anti fuzzy ideals of $\Gamma-$near-rings. P. Deena, G. Mohanraj and M. Akram have studied several properties of $T-$fuzzy ideals of rings and $T-$fuzzy ideals of near-rings. We extended the results of Akram to $\Gamma-$near-rings.

In this paper we define, characterize and study the $T-$fuzzy right and left ideals. Z. D. Wang introduced the basic concepts of TL-ideals. J. Prakashmanimaran, B. Chellappa and M. Jeyakumar introduced $T-$fuzzy right ideals of $\ell-$ ring. We introduced $T-$fuzzy right ideals of $\ell-$ near-ring. We compare fuzzy ideal introduced by Liu to $T-$fuzzy ideals. We have shown that ring is regular if and only if intersection of any $T-$fuzzy right ideal with $T-$fuzzy left ideal is equal to its product. We discuss some of its properties. We have shown that the meet of $T-$fuzzy ideal of $\ell-$ near-ring.

Definition: 1

A non-empty set R is called a near-ring with two binary operations "+" and " . " satisfying the following axioms:

(i) $(R, +)$ is a group

(ii) $(R, .)$ is a semigroup

(iii) $(x+y).z = x.z + y.z$, for all x, y, z in R (ie. Multiplicative is left distributive with respect to addition) We denote $x.y$ by xy.

Definition: 2

A non-empty set R is called lattice ordered near-ring or $\ell-$ near-ring if it has four binary operations "+", " $\cdot$ ", $\vee$, $\wedge$ defined on it and satisfy the following

(i) $(R, +)$ is a group

(ii) $(R, \cdot)$ is a semigroup

(iii) $(R, \vee, \wedge)$ is a lattice

(iv) $x.(y+z) = x.y + x.z$, for all x, y, z in R

(v) $x+(y\vee z) = (x+y)\vee(x+z)$; $x+(y\wedge z) = (x+y)\wedge(x+z)$

$(y\vee z)+x = (y+x)\vee(z+x)$; $(y\wedge z)+x = (y+x)\wedge(z+x)$

(vi) $x\cdot(y\vee z) = (xy)\vee(xz)$; $x\cdot(y\wedge z) = (xy)\wedge(xz)$

$(y\vee z)\cdot x = (yx)\vee(zx)$; $(y\wedge z)\cdot x = (yx)\wedge(zx)$,

for all x, y, z in R and $x \geq 0$

Example: 1

$(nZ, +, \cdot, \vee, \wedge)$ is a $\ell-$ near-ring, where Z is the set of all integers and $n \in Z$

Definition: 3

Let μ and λ be two fuzzy ideals of a $\ell-$near-ring R, then the sum $\mu+\lambda$ is a fuzzy set of R defined by

$$(\mu+\lambda)(x) = \begin{cases} \inf\left(\min\left(\mu(y),\ \lambda(z)\right)\right) & if\ \ x=y+z \\ 0 & otherwise \end{cases}, \quad \text{for all } x,\ y,\ z\in R.$$

Definition: 4

A fuzzy set λ of a $\ell-$near-ring R has the ***Infimum*** property if for any subset N of R, there exists a $a_0\in N$ such that $\lambda(a_0) = \inf_{a\in N}\lambda(a)$

Definition: 5

A mapping $T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is called a triangular norm [$t-$norm] if and only if it satisfies the following conditions:

(i). $T\ (x, 1) = T\ (1, x) = x$, for all $x \in [0,\ 1]$

(ii). $T\ (x,\ y) = T\ (y,\ x)$, for all x, y $\in [0,\ 1]$.

(iii). $T\ (x,\ T\ (y, z)) = T\ (T\ (x, y), z)$

(iv). $T\ (x,\ y) \le T\ (x,\ \text{z})$, whenever $y \le z$

Proposition: 1

The *minimum* $T-$norm ($\min T-$norm) is defined by $T(a,\ b) = \min\ \{a,\ b\}$

Some other $T-$norms are (i) $T_p\ (a,\ b) = ab$, (ii) $T_n(a,\ b) = \max\{a+b-1,\ 0\}$

and (iii) $T_w(a,\ b) = \begin{cases} a, & \text{if } \ b=1 \\ b, & \text{if } \ a=1 \\ 0, & \text{otherwise} \end{cases}$

Definition: 6

A fuzzy subset μ of a ring R is called $T-$fuzzy right (resp. left) ideal if

(i) $\mu(x-y) \ge T\ (\mu(x)\ ,\ \mu(y)) = \min\{\mu(x)\ ,\ \mu(y)\}$

(ii) $\mu(xy) \ge (\mu(x))$(resp. left $\mu(xy) \ge (\mu(y))$), for all x, y in R

Definition: 7

A fuzzy subset μ of a ring R is called $T-$fuzzy ideal if it is satisfied both right and left ideals. (Note: Every fuzzy right ideal of a ring R is a $T-$fuzzy right ideal).

Definition: 8

A fuzzy subset μ of a lattice ordered ring (or $\ell-$ring) R is called a fuzzy $\ell-$subring of R, if the following conditions are satisfied

(i) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$

(iii) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$

(iv) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all x, y in R

Example: 2

Consider the fuzzy subset μ of the $\ell-$ring $(Z, +, \cdot, \vee, \wedge)$; $\mu(x) = \begin{cases} 0.4 & if \ x \in \langle 4 \rangle \\ 0.9 & Z - \langle 4 \rangle \end{cases}$

Then μ is not a fuzzy $\ell-$subring. For example, let $x = 2$ and $y = 6$, then $x + y = 8$. Here $\mu(x) = 0.9$ and $\mu(y) = 0.9$. Therefore, $\min\{\mu(x), \mu(y)\} = \min\{0.9, 0.9\} = 0.9$. But $\mu(x+y) = 0.4$. Hence $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$. Thus μ is not fuzzy $\ell-$subring of R

Definition: 9

A fuzzy subset μ in a near-ring R is said to be a fuzzy subnear-ring of R if it satisfies the following conditions:

(i) $\mu(x - y) \geq \min(\mu(x), \mu(y))$, for all $x, y \in R$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in R$

Definition: 10

A fuzzy set μ in a near-ring R is said to be a fuzzy ideal of R, if the following conditions are satisfied,

(i) $\mu(x - y) \geq \min(\mu(x), \mu(y))$, for all x, y in R

(ii) $\mu(y + x - y) \geq \mu(x)$, for all x, y in R

(iii) $\mu(xy) \geq \mu(y)$; $\mu(xy) \geq \mu(x)$ for all $x, y \in R$

(iv) $\mu((x+z)y - xy) \geq \mu(z)$, for all $x, y, z \in R$

Definition: 11

A fuzzy subset μ in R is called fuzzy subnear-ring with respect to a $t-$norm of R if the following conditions are satisfied,

(i) $\mu(x-y) \geq T(\mu(x), \mu(y))$

(ii) $\mu(xy) \geq \mu(y); \mu(xy) \geq \mu(x)$, for all $x, y \in R$

Definition: 12

A fuzzy subset μ of a $T-$fuzzy near-ring R is called a $T-$fuzzy ideal, if the following conditions are satisfied,

(i) $\mu(x-y) \geq T(\mu(x), \mu(y))$, for all x, y in R

(ii) $\mu(y+x-y) \geq (\mu(x))$, for all x, y in R

(iii) $\mu(xy) \geq \mu(y)$, for all $x, y \in R$

(iv) $\mu((x+z)y-x y) \geq (\mu(z))$, for all $x, y, z \in R$

Proposition: 2

Every $T-$fuzzy ideal of a near-ring R is a $T-$fuzzy subnear-ring of R.Converse of Proposition 1 may not be true in general as seen in the following example.

Let $R=\{a, b, c, d\}$ be a set with binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

Then $(R, +, \bullet)$ is a near-ring. We define a fuzzy subset $\mu: R \to [0,1]$ by $\mu(a) > \mu(b) > \mu(d) = \mu(c)$. Let $T: [0,1] \times [0,1] \to [0,1]$ be a function defined by $T(x, y) = \max(x+y-1, 0)$, which is a $t-$norm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that μ is a $T-$fuzzy subnear-ring of R. It is clear that μ is also left $T-$fuzzy ideal of R. But μ is not $T-$fuzzy right ideal of R, since $\mu((c+d)d-cd) = \mu(d) < \mu(b)$

Definition: 13

A fuzzy subset μ of a $\ell-$near-ring R is called a $T-$fuzzy ideal, if the following conditions are satisfied,

(i) $\mu(x-y) \geq T(\mu(x), \mu(y))$

(ii) $\mu(y+x-y) \geq (\mu(x))$

(iii) $\mu(xy) \geq \mu(y)\,(or)\, \mu(xy) \geq \mu(x)$

(iv) $\mu((x+z)y-x\,y) \geq (\mu(z))$

(v) $\mu(x \vee y) \geq T(\mu(x), \mu(y))$

(vi) $\mu(x \wedge y) \geq T(\mu(x), \mu(y))$, for all $x, y, z \in R$

Example: 3

Now $(R=\{a, b, c\}, +, \cdot, \vee, \wedge)$ is a $\ell-$near-ring. The operations $+, \cdot, \vee$ and $\wedge$ defined by the following tables Consider the fuzzy subset μ of the $\ell-$near-ring R

$$\mu(x) = \begin{cases} 0.6 & if\ x=a \\ 0.5 & if\ x=b \\ 0.7 & if\ x=c \end{cases}$$

. Then μ is a $T-$fuzzy ideal of $\ell-$near-ring R

Theorem: 1

If $\{\mu_i(x), i \in I\}$ is a $T-$fuzzy ideal of a $\ell-$near-ring R then $\underset{i\in I}{\wedge} \mu_i$ is also a $T-$fuzzy ideal of $\ell-$near-ring R, where $\underset{i\in I}{\wedge} \mu_i$ is defined by $\left(\underset{i\in I}{\wedge} \mu_i\right)(x) = \inf\{\mu_i(x) : i \in I\}$, for all $x \in R$.

Proof:

Let $\{\mu_i(x), i \in I\}$ is a $T-$fuzzy ideal of a $\ell-$near-ring R and let $x, y, z \in R$

We have

(i) $$\underset{i\in I}{\wedge} \mu_i\ x-y = \inf\ \mu_i\ x-y\ \ | i\in I$$

$$\geq \inf\ T\ \mu_i\ x\ ,\ \mu_i\ y\ \ | i\in I$$

$$\geq T\ \inf\ \mu_i\ x\ \ | i\in I\ ,\ \inf\ \mu_i\ y\ \ | i\in I$$

$$= T\left(\underset{i\in I}{\wedge} \mu_i\ x\ ,\ \underset{i\in I}{\wedge} \mu_i\ y\right)$$

Therefore $\underset{i\in I}{\wedge} \mu_i\ x-y \geq T\left(\underset{i\in I}{\wedge} \mu_i\ x\ ,\ \underset{i\in I}{\wedge} \mu_i\ y\right)$, for any $x, y \in R$

(ii) Since $\mu(y+x-y) \geq \mu(x)$

We have $\underset{i\in I}{\wedge}\mu_i(y+x-y) = \inf\{\mu_i(y+x-y) \;\; | \, i\in I\}$

$$\geq \inf\{T\,(\mu_i(x),\, \mu_i(x)) \;\; | \, i\in I\}$$

$$\geq T\{\inf(\mu_i(x),\, \mu_i(x)) \;\; | \, i\in I\}$$

$$\geq T\{\inf(\mu_i(x)) \;\; | \, i\in I\}$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \;\; x\right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i(y+x-y) \geq \left(\underset{i\in I}{\wedge}\mu_i(x)\right)$, for all $x, y\in R$

(iii) Since $\mu(xy) \geq \mu(y)$ and $\mu(xy) \geq \mu(x)$

Let $x,\, y\in R$

Then $\underset{i\in I}{\wedge}\mu_i \;\; xy = \inf \;\; \mu_i \;\; xy \;\; | \, i\in I$

$$\geq \inf \;\; T \;\; \mu_i \;\; x \;\; | \, i\in I$$

$$= T \;\; \inf \;\; \mu_i \;\; x \;\; | \, i\in I$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \;\; x\right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i \;\; xy \geq \left(\underset{i\in I}{\wedge}\mu_i \;\; x\right)$, for any $x,\, y\in R$

And $\underset{i\in I}{\wedge}\mu_i \;\; xy = \inf \;\; \mu_i \;\; xy \;\; | \, i\in I$

$$\geq \inf \;\; T \;\; \mu_i \;\; y \;\; | \, i\in I$$

$$= T \;\; \inf \;\; \mu_i \;\; y \;\; | \, i\in I$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \;\; y\right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i \;\; xy \geq \left(\underset{i\in I}{\wedge}\mu_i \;\; y\right)$, for any $x,\, y\in R$

(iv) Since $\mu((x+z)y-x\,y) \geq \mu(z)$ and $\lambda((x+z)y-x\,y) \geq \lambda(z)$

$\underset{i\in I}{\wedge}\mu_i((x+z)y-x\,y) = \inf\{\mu_i((x+z)y-x\,y) \;\; | \, i\in I\}$

$$\geq \quad \inf\left\{T\left(\left(\left(x+z\right)y-x\ y\right),\left(\left(x+z\right)y-x\ y\right)\right) \quad | \ i\in I\right\}$$

$$= \quad T\left\{\inf\left(\left(\left(x+z\right)y-x\ y\right) \ | \ i\in I, \inf\left(\left(x+z\right)y-x\ y\right)\right) \quad | \ i\in I\right\}$$

$$= \quad T\left\{\inf\left(\mu_i\left(z\right), \mu_i\left(z\right)\right) \quad | \ i\in I\right\}$$

$$= \quad T\left\{\inf\left(\mu_i\left(z\right)\right) \quad | \ i\in I\right\}$$

$$= \quad \left(\wedge_{i\in I}\mu_i \ \ z\right)$$

Therefore $\wedge_{i\in I}\mu_i\left(\left(x+z\right)y-x\ y\right) \ \geq \ \left(\wedge_{i\in I}\mu_i\left(z\right)\right)$, for all $x, y, z\in R$

(v) $$\wedge_{i\in I}\mu_i \ \ x\vee y \quad = \quad \inf \ \ \mu_i \ \ x\vee y \quad | \ i\in I$$

$$\geq \ \inf \ \ T \ \ \mu_i \ \ x \ , \ \ \mu_i \ \ y \ \ | \ i\in I$$

$$\geq T \ \ \inf \ \ \mu_i \ \ x \ \ | \ i\in I \ , \ \inf \ \ \mu_i \ \ y \ \ | \ i\in I$$

$$= T\left(\wedge_{i\in I}\mu_i \ \ x \ \ , \ \wedge_{i\in I}\mu_i \ \ y \right)$$

Therefore $\wedge_{i\in I}\mu_i \ \ x\vee y \ \ \geq T\left(\wedge_{i\in I}\mu_i \ \ x \ \ , \ \wedge_{i\in I}\mu_i \ \ y\right)$, for any $x, \ y\in R$

(vi) $$\wedge_{i\in I}\mu_i \ \ x\wedge y \quad = \quad \inf \ \ \mu_i \ \ x\wedge y \quad | \ i\in I$$

$$\geq \ \inf \ \ T \ \ \mu_i \ \ x \ , \ \ \mu_i \ \ y \ \ | \ i\in I$$

$$\geq T \ \ \inf \ \ \mu_i \ \ x \ \ | \ i\in I \ , \ \inf \ \ \mu_i \ \ y \ \ | \ i\in I$$

$$= T\left(\wedge_{i\in I}\mu_i \ \ x \ \ , \ \wedge_{i\in I}\mu_i \ \ y \right)$$

Therefore $\wedge_{i\in I}\mu_i \ \ x\wedge y \ \ \geq T\left(\wedge_{i\in I}\mu_i \ \ x \ \ , \ \wedge_{i\in I}\mu_i \ \ y\right)$, for any $x, \ y\in R$

Hence $\wedge_{i\in I}\mu_i$ is a f $T-$fuzzy ideal of $\ell-$near-ring R

Theorem: 2

If $\{\mu_i(x) : i \in I\}$ is a family of T – fuzzy ideals of a ℓ – near-ring R, then $\underset{i\in I}{\wedge} \mu_i$ is also a T – fuzzy ideal of ℓ – near-ring R, where $\underset{i\in I}{\wedge} \mu_i$ is defined by

$$\left(\underset{i\in I}{\wedge} \mu_i\right) x = \inf \ \mu_i \ x \ : i \in I \ , \text{ for all } x \in R.$$

Proof:

To prove: $\underset{i\in I}{\wedge} \mu_i$ is a T – fuzzy ideals of R for a family of T – fuzzy ideal of ℓ – near-ring $\{\mu_i(x) : i \in I\}$. For any $x, y, z \in R$

(i) $$\underset{i\in I}{\wedge} \mu_i \ x - y \ = \ \inf \ \mu_i \ x - y \ \ | \, i \in I$$
$$\geq \inf \ \min \ \mu_i \ x \ , \ \mu_i \ y \ \ | \, i \in I$$
$$\geq \min \ \inf \ \mu_i \ x \ \ | \, i \in I \ , \ \inf \ \mu_i \ y \ \ | \, i \in I$$
$$= \min\left(\underset{i\in I}{\wedge} \mu_i \ x \ , \ \underset{i\in I}{\wedge} \mu_i \ y \right)$$

Therefore $\underset{i\in I}{\wedge} \mu_i \ x - y \ \geq \min\left(\underset{i\in I}{\wedge} \mu_i \ x \ , \ \underset{i\in I}{\wedge} \mu_i \ y \right)$, for any $x, y \in R$

(ii) Since $\mu(y + x - y) \geq \mu(x)$

We have $\underset{i\in I}{\wedge} \mu_i(y + x - y) = \inf\{\mu_i(y + x - y) \ | \, i \in I\}$
$$\geq \inf\{\min(\mu_i(x), \mu_i(x)) \ | \, i \in I\}$$
$$\geq \min\{\inf(\mu_i(x), \mu_i(x)) \ | \, i \in I\}$$
$$\geq \min\{\inf(\mu_i(x)) \ | \, i \in I\}$$
$$= \left(\underset{i\in I}{\wedge} \mu_i \ x \right)$$

Therefore $\underset{i\in I}{\wedge} \mu_i(y + x - y) \geq \left(\underset{i\in I}{\wedge} \mu_i(x)\right)$, for all $x, y \in R$

(iii) Since $\mu(x\,y) \geq \mu(y)$ and $\mu(x\,y) \geq \mu(x)$

Let $x,\ y \in R$

Then $\underset{i\in I}{\wedge}\mu_i \ \ xy \ = \inf \ \ \mu_i \ \ xy \ \ \ | \ i\in I$

$$\geq \inf \ \ \min \ \ \mu_i \ \ x \ \ \ | \ i\in I$$

$$= \min \ \inf \ \ \mu_i \ \ x \ \ \ | \ i\in I$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \ \ x \right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i \ \ xy \ \geq \left(\underset{i\in I}{\wedge}\mu_i \ \ x \right)$, for any $x,\ y \in R$

And $\underset{i\in I}{\wedge}\mu_i \ \ xy \ = \inf \ \ \mu_i \ \ xy \ \ \ | \ i\in I$

$$\geq \inf \ \ \min \ \ \mu_i \ \ y \ \ \ | \ i\in I$$

$$= \min \ \inf \ \ \mu_i \ \ y \ \ \ | \ i\in I$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \ \ y \right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i \ \ xy \ \geq \left(\underset{i\in I}{\wedge}\mu_i \ \ y \right)$, for any $x,\ y \in R$

(iv) Since $\mu\big((x+z)y - x\,y\big) \geq \mu(z)$ and $\lambda\big((x+z)y - x\,y\big) \geq \lambda(z)$

$\underset{i\in I}{\wedge}\mu_i\big((x+z)y - x\,y\big) = \inf\left\{\mu_i\big((x+z)y - x\,y\big) \ \ | \ i\in I\right\}$

$$\geq \inf\left\{\min\Big(\big((x+z)y - x\,y\big), \big((x+z)y - x\,y\big)\Big) \ \ | \ i\in I\right\}$$

$$= \min\left\{\inf\Big(\big((x+z)y - x\,y\big) \ | \ i\in I, \inf\big((x+z)y - x\,y\big)\Big) \ \ | \ i\in I\right\}$$

$$= \min\left\{\inf\big(\mu_i(z), \mu_i(z)\big) \ \ | \ i\in I\right\}$$

$$= \min\left\{\inf\big(\mu_i(z)\big) \ \ | \ i\in I\right\}$$

$$= \left(\underset{i\in I}{\wedge}\mu_i \ \ z \right)$$

Therefore $\underset{i\in I}{\wedge}\mu_i\big((x+z)y - x\,y\big) \geq \left(\underset{i\in I}{\wedge}\mu_i(z) \right)$, for all $x, y, z \in R$

(v) $$\wedge_{i\in I}\mu_i \;\; x\vee y \;\; = \;\; \inf \;\; \mu_i \;\; x\vee y \;\;\; | \, i\in I$$

$$\geq \;\inf \;\; \min \;\mu_i \;\; x \;,\;\; \mu_i \;\; y \;\;\; | \, i\in I$$

$$\geq \;\min \;\; \inf \;\; \mu_i \;\; x \;\;\; | \, i\in I \;,\; \inf \;\; \mu_i \;\; y \;\;\; | \, i\in I$$

$$= \;\min\left(\wedge_{i\in I}\mu_i \;\; x \;\;,\; \wedge_{i\in I}\mu_i \;\; y \;\right)$$

Therefore $\wedge_{i\in I}\mu_i \;\; x\vee y \;\; \geq \; \min \left(\wedge_{i\in I}\mu_i \;\; x \;\;,\; \wedge_{i\in I}\mu_i \;\; y \;\right)$, for any $x,\; y\in R$

(vi) $$\wedge_{i\in I}\mu_i \;\; x\wedge y \;\; = \;\; \inf \;\; \mu_i \;\; x\wedge y \;\;\; | \, i\in I$$

$$\geq \;\inf \;\; \min \;\mu_i \;\; x \;,\;\; \mu_i \;\; y \;\;\; | \, i\in I$$

$$\geq \;\min \;\; \inf \;\; \mu_i \;\; x \;\;\; | \, i\in I \;,\; \inf \;\; \mu_i \;\; y \;\;\; | \, i\in I$$

$$= \;\min\left(\wedge_{i\in I}\mu_i \;\; x \;\;,\; \wedge_{i\in I}\mu_i \;\; y \;\right)$$

Therefore $\wedge_{i\in I}\mu_i \;\; x\wedge y \;\; \geq \; \min \left(\wedge_{i\in I}\mu_i \;\; x \;\;,\; \wedge_{i\in I}\mu_i \;\; y \;\right)$, for any $x,\; y\in R$

Hence $\wedge_{i\in I}\mu_i$ is a family of $T-$fuzzy ideal of $\ell-$near-ring R

Theorem: 3

Let μ and λ be $T-$fuzzy ideal of a $\ell-$near-ring R. Then $\mu+\lambda$ is the smallest $T-$fuzzy ideal of a $\ell-$near-ring R containing both μ and λ.

Proof:

Let μ and λ be $T-$fuzzy ideal of a $\ell-$near-ring R and Let $x, y, z\in R$

Let $x = \; a+b \;\;,\;\; y = \; c+d$ and $a, b, c, d\in R$

Then $$x-y \;=\; a+b \;-\; c+d$$

$$= \; a+b \;\; -c-d$$

$$= \; a-c \;+\; b-d$$

$$= \; b+a-b \;\; -c+ \;\; c+b-c \;\; -d$$

$= e+f$, where $e = \; b+a-b \;\; -c$ and $f = \; c+b-c \;\; -d$

(i) We have $(\mu+\lambda)(x-y) = \bigvee_{x-y=e+f}\left[\mu(e)\wedge\lambda(f)\right]$

$$\geq \bigvee_{x=a+b,\ y=c+d}\left[T\left(\mu(b+a-b),\ \mu(c)\right)\ \wedge\ T\left(\lambda(c+b-c),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d}\left[T\left(\mu(a),\ \mu(c)\right)\ \wedge\ T\left(\lambda(b),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d}\left[T\left(\mu(a),\ \lambda(b)\right)\ \wedge\ T\left(\mu(c),\ \lambda(d)\right)\right]$$

$$= T\left(\left(\bigvee_{x=a+b}\left(\mu(a),\ \lambda(b)\right)\right)\wedge\left(\bigvee_{y=c+d}\left(\mu(c),\ \lambda(d)\right)\right)\right)$$

$$= T\left((\mu_A+\lambda_A)(x)\ ,\ (\mu_A+\lambda_A)(y)\right)$$

Therefore $(\mu+\lambda)(x-y)\ \geq\ T\left((\mu+\lambda)(x)\ ,\ (\mu+\lambda)(y)\right)$, for all $x, y\in R$

Now, for any $x = a+b$

Then $y+x-y = y+a+b-y$

$$= y+a-y+b$$

$$= y+a-y + y+b-y$$

$= c+d$, where $c = y+a-y$ and $d = y+b-y$

$y+x-y = c+d$, we have $x = -y+c+d+y$

$$= y+c-y+d$$

$$= y+c-y + y+d-y$$

(ii) Since $\mu(y+x-y)\ \geq\ \mu(x)$ and $\lambda(y+x-y)\ \geq\ \lambda(x)$

We have $(\mu+\lambda)(y+x-y) = \bigvee_{y+x-y=c+d}\left[\mu(c)\wedge\lambda(d)\right]$

$$\geq \bigvee_{x=a+b}\left[T\left(\mu(y+a-y)\right)\ \wedge\ \left(\lambda(y+b-y)\right)\right]$$

$$= \bigvee_{x=a+b}\left[T\left(\mu(a)\right)\ \wedge\ \left(\lambda(b)\right)\right]$$

$$= T\left((\mu+\lambda)(x)\right)$$

$$= (\mu+\lambda)(x)$$

Therefore $(\mu+\lambda)(y+x-y)\ \geq\ (\mu+\lambda)(x)$, for all $x, y\in R$

(iii) Since $\mu(xy) \geq \mu(x)$; $\lambda(xy) \geq \lambda(y)$, for all $x, y \in R$

Put $y = y_1 + y_2$, $y_1, y_2 \in R$

$$(\mu+\lambda)(xy) = (\mu+\lambda)\big(x\,(y_1+y_2)\big)$$
$$= (\mu+\lambda)(x\,y_1 + x\,y_2)$$
$$= \vee\left[\mu(x\,y_1) \wedge \lambda(x\,y_2)\right]$$
$$\geq \vee\left[T(\mu(x)) \wedge T(\lambda(x))\right]$$
$$\geq \vee\left[T(\mu(x), \lambda(x))\right]$$
$$= T((\mu+\lambda)(x))$$
$$= (\mu+\lambda)(x)$$

Therefore $(\mu+\lambda)(xy) \geq (\mu+\lambda)(x)$, for all $x, y \in R$

And

$$(\mu+\lambda)(xy) = (\mu+\lambda)\big((x_1+x_2)y\big)$$
$$= (\mu+\lambda)(x_1\,y + x_2\,y)$$
$$= \vee\left[\mu(x_1\,y) \wedge \lambda(x_2\,y)\right]$$
$$\geq \vee\left[T(\mu(y)) \wedge T(\lambda(y))\right]$$
$$\geq \vee\left[T(\mu(y), \lambda(y))\right]$$
$$= T((\mu+\lambda)(y))$$
$$= (\mu+\lambda)(y)$$

Therefore $(\mu+\lambda)(xy) \geq (\mu+\lambda)(y)$, for all $x, y \in R$

Now, for any $z = a+b$

Then we have

$$xy + (a+b)y - xy = xy + ay + by - xy$$
$$= xy + ay - xy + by$$
$$= xy + ay - xy + xy + by - xy$$

$= c + d$, where $c = xy + ay - xy$ and $d = xy + by - xy$

$xy + (a+b)y - xy = c + d$, we have

$$(a+b)y = -xy + c + d + xy$$
$$= xy + c - xy + d$$

$$= xy + c - xy + xy + d - xy$$

(vi) Since $\mu\left(\left(x+z\right)y - x\ y\right) \geq \mu\left(z\right)$ and $\lambda\left(\left(x+z\right)y - x\ y\right) \geq \lambda\left(z\right)$

We have $(\mu+\lambda)\left(\left(x+z\right)y - x\ y\right) = \bigvee_{(x+z)y-x\ y=c+d} \left[\mu(c) \wedge \lambda(d)\right]$

$$\geq \bigvee_{z=a+b} \left[T\left(\mu\left(xy+c-xy\right)\right) \wedge \left(\lambda\left(xy+d-xy\right)\right)\right]$$

$$= \bigvee_{z=a+b} \left[T\left(\mu(z)\right) \wedge \left(\lambda(z)\right)\right]$$

$$= T\left(\mu(z)\right) \wedge \left(\lambda(z)\right)$$

$$= (\mu+\lambda)(z)$$

Therefore $(\mu+\lambda)(y+x-y) \geq (\mu+\lambda)(z)$, for all $x, y, z \in R$

(v) We have $(\mu+\lambda)(x \vee y) = \bigvee_{x-y=e+f} \left[\mu(e) \wedge \lambda(f)\right]$

$$\geq \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(b+a-b),\ \mu(c)\right) \wedge T\left(\lambda(c+b-c),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(a),\ \mu(c)\right) \wedge T\left(\lambda(b),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(a),\ \lambda(b)\right) \wedge T\left(\mu(c),\ \lambda(d)\right)\right]$$

$$= T\left(\left(\bigvee_{x=a+b} \left(\mu(a),\ \lambda(b)\right)\right) \wedge \left(\bigvee_{y=c+d} \left(\mu(c),\ \lambda(d)\right)\right)\right)$$

$$= T\left(\left(\mu_A + \lambda_A\right)(x),\ \left(\mu_A + \lambda_A\right)(y)\right)$$

Therefore $(\mu+\lambda)(x \vee y) \geq T\left((\mu+\lambda)(x),\ (\mu+\lambda)(y)\right)$, for all $x, y \in R$

(vi) We have $(\mu+\lambda)(x \wedge y) = \bigvee_{x-y=e+f} \left[\mu(e) \wedge \lambda(f)\right]$

$$\geq \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(b+a-b),\ \mu(c)\right) \wedge T\left(\lambda(c+b-c),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(a),\ \mu(c)\right) \wedge T\left(\lambda(b),\ \lambda(d)\right)\right]$$

$$= \bigvee_{x=a+b,\ y=c+d} \left[T\left(\mu(a),\ \lambda(b)\right) \wedge T\left(\mu(c),\ \lambda(d)\right)\right]$$

$$= T\left(\left(\bigvee_{x=a+b} \left(\mu(a),\ \lambda(b)\right)\right) \wedge \left(\bigvee_{y=c+d} \left(\mu(c),\ \lambda(d)\right)\right)\right)$$

$$= T\left(\left(\mu_A + \lambda_A\right)(x),\ \left(\mu_A + \lambda_A\right)(y)\right)$$

Therefore $(\mu+\lambda)(x\wedge y) \geq T((\mu+\lambda)(x), (\mu+\lambda)(y))$, for all $x, y\in R$

Thus $\mu+\lambda$ is the smallest $T-$fuzzy ideal of a $\ell-$near-ring R containing both μ & λ.

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