International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 6, Number 1 (2014), pp. 11-19

http://www.irphouse.com

$\left|\left(\overline{N}, p_n\right) - B\right|$-Summability of a Factored Fourier Series

Shubhra Sharma[1] and Chitaranjan Khadanga[2]

Dept. of Mathematics Rungta College of Engg.& Tech,.Bhilai,(C.G) India
Email: shubhrasharma13@gmail.com
Dept.of Mathematics Rungta College of Engg.& Tech,.Bhilai,(C.G) India
Email: chitakhadanga@gmail.com

Abstract

In this paper, we have been proved an analogue theorem on $\left|\left(\overline{N}, p_n\right) - B\right|$ summability of a factored Fourier series.

Key Words: Summability factor, Banach Summability, Infinite Series, Norlund-riesz Sumability.

INTRODUCTION

Let $\{s_n\}$ be the sequence of partial sums of a series $\sum a_n$. Let the sequence $\{t_k(n)\}$ be defined by

(1.1) $$t_k(n) = \frac{1}{k}\sum_{v=0}^{k-1} s_{n+v},\ k \in N$$

Then $t_k(n)$ is said to be the k-th element of the Banach transformed sequence. If

(1.2) $$\lim_{k\to\infty} t_k(n) = s \text{, a finite number,}$$

Uniformly for all $n \in N$, then $\sum a_n$ is said to be Banach summable to s, Further, if the series

(1.3) $$\sum_{k=1}^{\infty}\left|t_n^{(n)} - t_{k+1}^{(n)}\right| < \infty$$

Uniformly for all $n \in N$, then the series $\sum a_n$ is said to be absolutely Banach summable or simply $|B|$-summable.

Let $\{p_n\}$ be a sequence of non-negative numbers with

$P_n = \sum_{v=1}^{n} p_v$, $n \in N$, and $p_0 = 0,\ p_{-i} = 0,\ i = 1,2,...$

Let the sequence $\{T_k(n)\}$ be defined by

(1.4) $T_k(n) = \frac{1}{P_k} \sum_{v=1}^{k} p_v\ s_{n+v-1},\ k \in N$

Then $T_k(n)$ is said to be the k-th element of the $(\overline{N}, p_n)$-Banach transformed sequence. If

(1.5) $\lim_{k\to\infty} T_k(n) = s$, a finite number

Uniformly for all $n \in N$, then the series $\sum a_n$ is said to be $(\overline{N}, p_n) - B$-summable. Further, if the series

(1.6) $\sum_{k=1}^{\infty} |T_k(n) - T_{k+1}(n)| < \infty$

Uniformly for all $n \in N$, then the series $\sum a_n$ is said to be absolutely $(\overline{N}, p_n) - B$ summable or simply $(\overline{N}, p_n) - B$-summable.

In case if $p_n = 1,\ \forall n \in N$, then $|(\overline{N}, p_n) - B|$-summability reduces to $|B|$-summability.

2. Let $\sum_{n=s}^{\infty} A_n(x)$ be the Fourier series of a 2π-periodic function $f(t)$ and *L*-integrable in $(-\pi, \pi)$. Then

(2.1) $A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\ dt,\ \ n = 0,1,2,...$

where $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$.

Dealing with absolute summability of factored Fourier series, Chow established that if $\{\lambda_n\}$ is a non-negative convex sequence and the series $\sum \frac{\lambda_n}{n}$ converges, then the factored Fourier series $\sum \lambda_n\ A_n(x)$ of *f* is summable $|c,1|$ for almost all values of *x*.

KNOWN RESULT

If $\phi(t)\log\left(\frac{k}{t}\right) \in BV(0,\pi)$, then the factored Fourier series $\sum \lambda_n A_n(x)$ is $|B|$-summable, for $\{\lambda_n\}$ to be a non-negative convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$.

3. MAIN THEOREM:

If $\phi(t)\log\left(\frac{k}{t}\right) \in BV(0,\pi)$, then the factored Fourier series $\sum \lambda_n \ A_n(x)$ is $\left|\left(\overline{N}, p_n\right) - B\right|$- summable, for $\{\lambda_n\}$ to be non-negative convex sequence and $\{p_n\}$ to be a sequence of non-negative number with $P_n = \sum_{k=1}^{n} p_k \neq 0$ such that

(i) $\sum \frac{\lambda_n}{n} < \infty$

and

(ii) $np_n = O(P_n)$.

4. REQUIRED LEMMAS

We require the following lemmas for proof of the above main theorem.

Lemma-1 If $\{\lambda_n\}$ is a positive convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$, then $\{\lambda_n\}$ is monotonically decreasing.

Lemma-2 Let $\{p_n\}$ be a positive non-decreasing sequence. Let $\tau = \left[\frac{1}{t}\right]$, then

$$\sum_{n=a}^{b} p_n \ \cos\ nt = O(P_\tau)$$

$$\sum_{n=a}^{b} p_n \ \sin\ nt = O(P_\tau)$$

Lemma-3 Let $\{T_k(n)\}$ be as defined in (1.4), then

$$T_k(n) - T_{k+1}(n) = -\frac{p_{k+1}}{P_k \ P_{k+1}} \sum_{v=0}^{k} P_v \ a_{n+v} \ .$$

Proof: We have

$$T_k(n) = \frac{1}{P_k} \sum_{v=1}^{k} p_v \ s_{n+v-1}$$

$$= \frac{1}{P_k} \sum_{v=1}^{k} \ p_v \sum_{i=1}^{n+v-1} a_i = \frac{1}{P_k}\left\{\sum_{i=1}^{n} a_i \ \sum_{v=1}^{k} p_v \ + \sum_{i=n}^{n+k-1} a_i \ \sum_{v=i-n+1}^{k} p_v\right\}$$

$$= \frac{1}{P_k} \ P_k \ s_n \ + \sum_{i=n}^{n+k-1} \left(P_k - P_{i-n}\right) a_i \ .$$

Thus

$$T_k(n) - T_{k+1}(n) = \frac{1}{P_k} \sum_{i=n}^{n+k-1} \left(P_k - P_{i-n}\right) a_i - \frac{1}{P_{k+1}} \sum_{i=n}^{n+k} \left(P_{k+1} - P_{i-n}\right) a_i$$

$$= -\frac{p_{n+1}}{P_k\, P_{n+1}} \sum_{i=n}^{n+k-1} P_{i-n}\, a_i - \frac{p_{k+1}}{P_{k-1}}\, a_{n+k}$$

$$= -\frac{p_{k+1}}{P_k P_{k+1}} \sum_{i=n}^{n+k} P_{i-n}\, a_i = -\frac{p_{k+1}}{P_k\, P_{k+1}} \sum_{v=0}^{k} P_v\, a_{n+v}\,.$$

PROOF OF MAIN THEOREM

Part-i

We have $A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\; dt$

$$= \frac{2}{\pi} \int_0^{\pi} \left\{ \phi(t) \log\left(\frac{k}{t}\right) \right\} \frac{\cos nt}{\log\left(\frac{k}{t}\right)}\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} h(t) \frac{\cos nt}{\log\left(\frac{k}{t}\right)}\, dt, \; where\;\; h(t) = \phi(t) \log\left(\frac{k}{t}\right)$$

Thus

$$A_n(x) = 0\left(\frac{1}{n(\log n)^2}\right) - \frac{2}{\pi} \int_0^{\pi} dh(t) \left\{ \left(\log\left(\frac{k}{t}\right)\right)^{-1} \frac{\sin\; nt}{n} \right\}.$$

Now, for the factored series $\sum \lambda_n A_n(x)$, we have

$$\sum_{k=1}^{\infty} \left| T_k(n) - T_{k+1}(n) \right| = \sum_{k=1}^{\infty} \frac{p_{k+1}}{P_k P_{k+1}} \left| \sum_{v=0}^{k} P_v\;\; \lambda_{n+v}\; A_{n+v}^{(x)} \right|$$

$$= \sum_{k=1}^{\infty} \frac{p_{n+1}}{P_n P_{n+1}} \left| \sum_{v=0}^{k} P_v\, \lambda_{n+v} \left\{ 0\left(\frac{1}{(\log(n+v))}\right)^2 - \frac{2}{\pi} \int_0^{\pi} dh(t) \left\{ \left(\log\left(\frac{k}{t}\right)\right)^{-1} \frac{\sin(n+v)t}{n+v} \right\} \right\} \right|$$

$$\leq \sum_{k=1}^{\infty} \frac{p_{n+1}}{P_k P_{k+1}} \left| \sum_{v=0}^{k} P_v\, \lambda_{n+v} \cdot 0\left(\frac{1}{(n+v)(\log(n+v))^2}\right) \right|$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{p_{k+1}}{P_k P_{k+1}} \left| \sum_{v=0}^{k} P_v\, \lambda_{n+v} \left(\log\left(\frac{k}{t}\right)\right)^{-1} \frac{\sin(n+v)t}{n+v} \int_0^{\pi} dh(t) \right|$$

$= S_1 + S_2$, say

Now,

$$S_1=0(1)\sum_{k=1}^{\infty}\frac{p_{k+1}}{P_kP_{k+1}}\left|\sum_{v=0}^{k}\frac{P_v\ \lambda_{n+v}}{(n+v)(\log(n+v))^2}\right|$$

$$=0(1)\sum_{v=1}^{\infty}\frac{P_v\ \lambda_{n+v}}{(n+v)(\log(n+v))^2}\sum_{k=v}^{\infty}\left(\frac{1}{P_k}-\frac{1}{P_{k+1}}\right)$$

$$=0(1)\sum_{v=1}^{\infty}\frac{1}{(n+v)\ (\log(n+v))^2}<\infty\text{, uniformly in } n.$$

Further,

$$S_2=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|\sum_{v=0}^{k}P_v\ \lambda_{n+v}\left(\log\left(\frac{k}{t}\right)\right)^{-1}\frac{\sin(n+v)t}{n+v}\int_0^{\pi}dh(t)\right|$$

$$=0(1)\sum_{k=1}^{\infty}\frac{p_{n+1}}{P_k\ P_{n+1}}\left|\sum_{v=0}^{k}\frac{P_v\ \lambda_{n+v}\left(\log\left(\frac{k}{t}\right)\right)^{-1}\sin(n+v)\,t}{n+v}\right|,\ as\ h(t)\in BV(0,\pi)$$

$$=0(1)\ (\log\ \tau)^{-1}\ (S_{21}+S_{22})\text{, say.}$$

Now,

$$S_{21}=\sum_{k=1}^{\tau}\frac{p_{n+k}}{P_k\ P_{n+1}}\left|\sum_{v=0}^{k}\frac{P_v\ \lambda_{n+v}\ \sin(n+v)\,t}{n+v}\right|$$

$$=0(t)\sum_{k=1}^{\tau}\frac{p_{n+1}}{P_kP_{n+1}}\sum_{v=0}^{k}P_v$$

$$=0(t)\sum_{k=1}^{\tau}\frac{k\ p_{k+1}}{P_{k+1}}$$

$$=0(t)\cdot 0(\tau)=0(\log\tau).$$

Again

$$S_{22}=\sum_{k=\tau+1}^{\infty}\frac{p_{k+1}}{P_k\ P_{k+1}}\left|\sum_{v=0}^{k}\frac{P_v\ \lambda_{n+v}}{n+v}\sin(n+v)\,t\right|$$

$$=\sum_{k=\tau+1}^{\infty}\frac{p_{k+1}}{P_k\ P_{n+1}}\left|\sum_{v=0}^{k-1}\Delta\left(\frac{P_v\ \lambda_{n+v}}{n+v}\right)\sum_{p=1}^{v}\sin(n+p)\,t+\frac{P_k\ \lambda_{n+k}}{n+k}\sum_{p=1}^{k}\sin(n+p)\,t\right|$$

$$=0(\tau)\sum_{k=\tau+1}^{\infty}\frac{p_{k+1}}{P_k\ P_{n+1}}\left|\sum_{v=0}^{k-1}\left(\frac{-p_{v+1}\ \lambda_{n+v}}{n+v}+\frac{P_{v+1}\ \Delta\lambda_{n+v}}{n+v}+\frac{P_{v+1}\ \lambda_{n+v+1}}{(n+v)\,(n+v+1)}\right)+\frac{P_k\ \lambda_{n+k}}{n+k}\right|$$

$$\le 0(\tau)\left[\sum_{k=\tau+1}^{\infty}\frac{p_{k+1}}{P_kP_{n+1}}\sum_{v=0}^{k-1}\left\{\frac{p_{v+1}\ \lambda_{n+v}}{n+v}+\frac{P_{v+1}\ \Delta\lambda_{n+v}}{n+v}+\frac{P_{v+1}\ \lambda_{n+v+1}}{(n+v)(n+v+1)}\right\}\times\frac{P_k\ \lambda_{n+k}}{n+k}\right]$$

$$=S_{221}+S_{222}+S_{223}+S_{224},$$

Now

$$S_{221} = 0(\tau) \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k P_{n+1}} \sum_{v=0}^{k-1} \frac{p_{v+1}\ \lambda_{n+v}}{n+v}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \lambda_{n+v}}{n+v} \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k\ P_{k+1}} + \sum_{k=\tau+1}^{\infty} \frac{p_{v+1}\ \lambda_{n+v}}{n+v} \sum_{k=v+1}^{\infty} \frac{p_{k+1}}{P_k\ P_{n+1}}\right\}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \lambda_{n+v}}{n+v\ P_{\tau+1}} + \sum_{v=\tau}^{\infty} \frac{p_{v+1}\ \lambda_{n+v}}{(n+v)\ P_{v+1}}\right\}$$

$$= 0(\tau)\left\{0\left(\tau^{-1}\right)\sum_{v=1}^{\tau} \frac{\lambda_{n+v}}{n+v}\ 0\left(\tau^{-1}\right) \sum_{v=\tau}^{\infty} \frac{\lambda_{n+v}}{n+v}\right\}$$

$$= 0(1) \sum_{v=1}^{\infty} \frac{\lambda_{n+v}}{n+v}$$

$$= 0(\log\ \tau).$$

$$S_{222} = 0(\tau) \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k P_{n+1}} \sum_{v=0}^{k-1} \frac{p_{v+1}\ \Delta\lambda_{n+v}}{n+v}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \Delta\lambda_{n+v}}{n+v} \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k\ P_{k+1}} + \sum_{v=\tau}^{\infty} \frac{p_{v+1}\ \Delta\lambda_{n+v}}{n+v} \sum_{k=v+1}^{\infty} \frac{p_{k+1}}{P_k\ P_{n+1}}\right\}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \Delta\lambda_{n+v}}{(n+v)\ P_{\tau+1}} + \sum_{v=\tau}^{\infty} \frac{p_{v+1}\ \Delta\lambda_{n+v}}{(n+v)\ P_{v+1}}\right\}$$

$$= 0(\tau)\left\{0\left(\tau^{-1}\right)\sum_{v=1}^{\tau} \Delta\lambda_{n+v} + 0\left(\tau^{-1}\right) \sum_{v=\tau}^{\infty} \Delta\lambda_{n+1}\right\}$$

$$= 0(1) \sum_{v=1}^{\tau} \Delta\lambda_{n+v}$$

$$= 0(\log\ \tau).$$

Also,

$$S_{223} = 0(\tau) \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k P_{n+1}} \sum_{v=0}^{k-1} \frac{p_{v+1}\ \lambda_{n+v+1}}{(n+v)(n+v+1)}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \lambda_{n+v+1}}{(n+v)(n+v+1)} \sum_{k=\tau+1}^{\infty} \frac{p_{k+1}}{P_k\ P_{k+1}} + \sum_{v=\tau}^{\infty} \frac{p_{v+1}\ \lambda_{n+v+1}}{(n+v)(n+v+1)} \sum_{k=v+1}^{\infty} \frac{p_{n+1}}{P_k\ P_{n+1}}\right\}$$

$$= 0(\tau)\left\{\sum_{v=1}^{\tau} \frac{p_{v+1}\ \lambda_{n+v+1}}{(n+v)\ (n+v+1)P_{\tau+1}} + \sum_{v=\tau}^{\infty} \frac{\lambda_{n+v+1}}{(n+v)\ (n+v+1)}\right\}$$

$$= 0(\tau)\left\{0\left(\tau^{-1}\right)\sum_{v=1}^{\tau} \frac{\lambda_{n+v+1}}{n+v+1} + 0\left(\tau^{-1}\right) \sum_{v=\tau}^{\infty} \frac{\lambda_{n+v+1}}{n+v+1}\right\}$$

$$= 0(1) \sum_{v=1}^{\tau} \frac{\lambda_{n+v+1}}{n+v+1}$$

$$= 0(\log\ \tau).$$

Finally,

$$S_{224} = 0(\tau) \sum_{k=\tau+1}^{\infty} \frac{p_{n+1}}{P_k P_{n+1}} \cdot \frac{P_k \ \lambda_{n+k}}{n+k}$$

$$= 0(\tau) \sum_{k=\tau+1}^{\infty} \frac{p_{n+1} \ \lambda_{n+k}}{(n+k)\, P_{k+1}}$$

$$= 0(\tau)\, 0\left(\tau^{-1}\right) \sum_{k=\tau+1}^{\infty} \frac{\lambda_{n+k}}{n+k}$$

$$= O(\log \tau) \ .$$

Then $S_2 = O(1)$.

Hence $\therefore \sum_{n=1}^{\infty} \left|T_k(n) - T_{n+1}(n)\right| < \infty$, uniformly in *n*.

Part-ii

If $\phi(t) \in BV(0,\pi)$, then the Fourier Series $\sum A_n(x)$ is $\left|\left(\overline{N}, p_n\right)-B\right|$-summable, for $\{p_n\}$ to be a sequence of non-negative number with $P_n = \sum_{k=1}^{n} p_n \neq 0$ such that $np_n = O(P_n)$

PROOF:

We have $\left|T_k(n) - T_{k+1}(n)\right| = \dfrac{p_{n+1}}{P_k + P_{n+1}} \left|\sum_{v=0}^{k} P_v \ A_{n+v}\right|$

$$= \frac{2}{\pi} \ \frac{p_{n-1}}{P_k P_{n-1}} \left| \sum_{v=0}^{k} \int_0^{\pi} \frac{P_v}{n+v} \sin(n+v) + \ d\phi(t) \right|$$

$$\therefore \sum_{k=1}^{\infty} \left| T_k(n) - T_{n+1}(n) = \frac{2}{\pi} \sum_{v=0}^{\infty} \frac{P_v}{n+v} \left| \sum_{v=0}^{k} \frac{P_v}{n+v} \sin\,(n+N)\, t \right| \right|$$

Now

$$\sum_{k=1}^{\tau} \frac{p_{n+1}}{P_n \ P_{n+1}} \left| \sum_{v=0}^{k} \frac{P_v}{n+v} \sin(n+v)t \right|$$

$$= \sum_{k=1}^{\tau} \frac{p_{n+1}}{P_n \ P_{n+1}} \left| \sum_{v=0}^{k} \frac{P_v\,(n+v)}{n+v} \ t \right|$$

$$= 0(t) \sum_{n=1}^{\tau} \frac{p_{n+1}}{P_n P_{n+1}} (n+v)\, P_k = c(t) \sum_{k=1}^{\tau} \frac{p_{n+1}}{P_{n+1}} = 0(t) \cdot 0(\tau) = 0(1)$$

Finally,

$$\sum_{k>\tau} \frac{p_{k+1}}{P_k \ P_{n+1}} \left| \sum_{v=0}^{k} \frac{P_v}{n+v} \sin(n+v) \ t \right|$$

$$= \sum_{n>\tau} \frac{p_{n+1}}{P_n P_{n+1}} \cdot \frac{P_n}{n+n} \left| \sum_{v=0}^{n} \sin(n+v)\, t \right| \therefore \left\{ \frac{P_v}{n+v} \right\}^v$$

$$= \sum_{k>\tau} \frac{p_{k+1}}{(n+k)P_{n+1}}\, 0(\tau)$$

$$= 0(\tau) \sum_{k>\tau} \frac{1}{(n+n)}\, (k+1)$$

$$= c(T) \cdot c\left(\tau^{-1}\right) = 0(1)$$

$$\therefore \sum_{n=1}^{\infty} \left| T_k(n) - T_{n+1}(n) \right| < \infty, \text{ uniformly in } t.$$

If $\sum_{r=1}^{\infty} \left(\frac{P_r}{p_r} \right)^{k-1} \left| T_r(n) - T_{r+1}(n) \right|^k < \infty$, uniformly in n.

Then $\sum a_n$ is said to $\left| \left(\overline{N}, p_n \right) - B \right|$ summable.

This Complete the prove the theorem.

THEOREM-2: If $\phi(t) \in BV(0,\pi)$, then the Fourier series $\sum A_n(x)$ is $\left| \left(\overline{N},\ p_n \right) - B \right|_k$ -summable $k \geq 1$ for $\{p_n\}$ to be sequence with $P_n = \sum_{n=1}^{n} p_n \neq 0$ such that (i) $np_n = 0(P_n)$, (ii) $P_n = 0(np)$.

PROOF: We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\ nt$$

$$= \frac{2}{n\pi} \int_0^{\pi} \sin\ nt\ d\phi(t),\ n = 1,2,3,\ldots$$

Then

$$\sum_{r=1}^{\infty} \left(\frac{P_r}{p_r} \right)^{k-1} \left| T_r(n) - T_{r+1}(n) \right|^k = \frac{2}{\pi} \sum_{v=1}^{\infty} \left(\frac{P_v}{p_v} \right)^{k-1} \left(\frac{p_{r+1}}{P_v\ P_{v+1}} \right)^k \left| \sum_{v=1}^{r} \int_0^{\pi} \frac{P_v}{n+v} \sin(n+v)\ t\ \phi(t) \right|^k$$

$$= 0(1) \left(\sum_{r=1}^{\tau} + \sum_{r>z} \right) = \sum_1 + \sum_2 \cdots,\ \phi(t) \in BV(0,\pi)$$

$$\sum_1 = \sum_{r=1}^{\tau} \left(\frac{P_v}{p_v} \right)^{k-1} \left(\frac{p_{v+1}}{P_v P_{v+1}} \right)^k \left| \sum_{v=1}^{r} \frac{P_v}{n+v} \sin(n+v)\ t \right|^k$$

$$= \sum_{v=1}^{\tau} \left(\frac{P_v}{p_r} \right)^{k-1} \left(\frac{p_{v+1}}{P_v\ P_{v+1}} \right)^k \left| \sum_{v=1}^{r} \frac{P_v(n+v)t}{n+v} \right|^t$$

$$= 0(t^k)\sum_{v=1}^{\tau}\left(\frac{P_v}{p_v}\right)^{n-1}\left(\frac{P_{v+1}}{P_v\ P_{v+1}}\right)^k r^k\ P_r^k = 0\left(t^k\right)\sum_{v=1}^{\tau}\frac{r^{k-1}r^k}{(r+1)^k}$$

$$= 0\left(t^k\right)\sum_{v=1}^{\tau} r^{k-1} = 0\left(t^k\right) 0\left(\tau^k\right) = 0(1)$$

$$\sum_2 = \sum_{v>\tau}\left(\frac{P_v}{p_v}\right)^{k-1}\left(\frac{p_{v+1}}{P_v\ P_{v+1}}\right)^k\left(\sum_{v=1}^{r}\frac{P_v}{n+v}\sin(n+v)\ t\right)^k$$

$$= \sum_{r>\tau}\left(\frac{P_r}{p_r}\right)^{k-1}\left(\frac{p_{r+1}}{P_v P_{v+1}}\right)^k\left(\frac{P_v}{n+v}\right)^k 0\left(\tau^k\right)$$

$$= 0\left(\tau^k\right)\sum_{r>\tau}\frac{r^{k-1}}{(n+v)^k(v+1)^k} = c\left(\tau^k\right)\sum_{v>\tau}\frac{1}{v(n+r)^k} = d\left(\tau^k\right)\cdot 0\left(\tau^{-k}\right)$$

$$= 0(1)$$

References:

[1] ABEL, N. H : Üntersuchungen über die Reihe $1 + mx + \frac{m(m-1)}{2}x^2 + \cdots$,*Journal fur reine and angewants mathematik* (crelle) I(1826), 311-339.

[2] BANACH, S : Theorie des Operations Lineaires, *Monoyrafie Mathematyezne*, Vol.1, Warsaw (1932).

[3] BOR, H : On the local property of $\left|\overline{N}, p_n\right|_k$ summability of factored Fourier series, *Journal of Mathematical Analysis and applications*, 163, 220-226 (1992).

[4] CHAMPMAN, S and HARDY, G.H : A general view of the theory at summable series, *Quarterly Journal of Mathematics*, 42(1911), 181-215..

[5] CHAMPMAN, S : On non-integral orders of summability of series and integrals, *Proceedings of the London Mathematical Society*, (2), 9(1910) 369-409.

[6] CHOW, A.C : Note on convergence and summability factors, Journal of the London Mathematical Society, 29(19054),459 – 476.

[7] MISRA, M., MISRA, M., RAUTO, K. : Absolute Banach Summabilty of Fourier Series, International Journal of Mathematical Sciences, June 2006, Vol. 6 No.1, 39 – 45.