Intuitionistic Fuzzy Ideals in Hyper BCI-Algebras

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Abstract

In this paper we introduce distributive hyper BCI-ideals and the concept of intuitionistic fuzzy set is applied to investigate the relations between intuitionistic fuzzy distributive hyper BCI-ideals and the distributive hyper BCI-ideals of a hyper BCI-algebra.

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Introduction

The hyper structure theory was introduced by F. Marty [13] in 1934. Y.B. Jun et al. applied this concept to BCK-algebras [7] and Xiao Xin Long introduced a hyper BCI-algebra [12], as generalization of a BCI-algebra. Different types of hyper BCI-ideals are also defined in [12]. The proper hyper BCI-algebras which coincide with hyper BCK-algebras and p-semisimple BCI-algebras of order 3 are investigated in [5]. In [8-10] the fuzzification of hyper BCK-ideals is discussed and the related results are developed. The notion of Bi-polar-valued fuzzy hyper subalgebra (briefly BFHS) of a hyper BCI-algebra based on Bi-polar-valued fuzzy set and related properties are established in [6]. Further, Bi-polar-valued fuzzy characteristic hyper subalgebra is also stated.
The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy sets. After that many researchers considered the intuitionistic fuzzification of ideals and subalgebras in BCK/BCI-algebras. In this paper we introduce distributive hyper BCI-ideals and the concept of intuitionistic fuzzy set is applied to investigate the relations between intuitionistic fuzzy distributive hyper BCI-ideals of a hyper BCI-algebra.

Preliminaries
By a BCI-algebra we mean an algebra \((X, *, 0)\) type \((2, 0)\) satisfying the axioms:

(i) \(((x * y) * (x * z)) * (z * y) = 0, \)
(ii) \((x * (x * y)) * y = 0, \)
(iii) \(x * x = 0, \)
(iv) \(x * y = 0\) and \(y * x = 0\) imply \(x = y, \) for all \(x, y, z \in X.\)

If a BCI-algebra \(X\) satisfies the following identity:
(v) \(0 * x = 0,\) for all \(x \in X,\)
then \(X\) is called a BCK-algebra. Any BCI/BCK-algebra \(X\) satisfies the following axioms:
(vi) \(x * 0 = x,\)
(vii) \(x \leq y\) imply \(x * z \leq y * z\) and \(z * y \leq z * x,\)
(viii) \((x * y) * z = (x * z) * y,\)
(ix) \((x * z) * (y * z) \leq x * y, \) for all \(x, y, z \in X.\)

In a BCI-algebra, we can define a partial ordering "\(\leq\)" by \(x \leq y\) if and only if \(x * y = 0.\) A subset \(S\) of a BCK/BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S.\) An ideal of a BCK/BCI-algebra \(X\) is a subset \(I\) of \(X\) containing \(0\) such that if \(x * y \in I\) and \(y \in I\) then \(x \in I.\) Note that every ideal \(I\) of a BCK/BCI-algebra \(X\) has the following property:
\(x \leq y\) and \(y \in I\) imply \(x \in I.\)

In what follows, \(X\) will denote a BCI-algebra unless otherwise specified.

An intuitionistic fuzzy set \(A\) in a non-empty set \(X\) is an object having the form
\(A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},\)
where the functions \(\mu_A : X \rightarrow [0,1]\) and \(\nu_A : X \rightarrow [0,1]\) denote the degree of membership (namely \(\mu_A(x)\)) and the degree of non membership (namely \(\nu_A(x)\)) of each element \(x \in X\) to the set \(A\) respectively, and \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\) for all \(x \in X.\)

Such defined objects are studied by many authors (see for Example two journals:
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1. Fuzzy Sets and 2. Notes on Intuitionistic Fuzzy Sets) and have many interesting applications not only in mathematics (See Chapter 5 in the book [2]).

Let \( X \) be a non empty set endowed with a hyper operation "\( \circ \)" that is "\( \circ \)" is a function from \( X \times X \) to \( P^*(X) = P(X) - \{ \emptyset \} \). For two subsets \( A \) and \( B \) of \( X \), denote by \( A \circ B \) the set

\[
\cup_{a \in A, b \in B} a \circ b
\]

**Definition 2.1** [14]: By a fuzzy set \( \mu \) in \( X \), we mean a function \( \mu : X \to [0,1] \).

**Definition 2.2** [12]: By a hyper BCI-algebra, it is meant a nonempty set \( X \) endowed with a hyper operation "\( \circ \)" and a constant 0 satisfying the following axioms:

(I) \((x \circ z) \circ (y \circ z) \ll x \circ y,

(II) \((x \circ y) \circ z = (x \circ z) \circ y,

(III) x \ll x,

(IV) x \ll y and y \ll x imply x = y,

(V) 0 \circ (0 \circ x) \ll x, x \neq 0,

for all \( x, y, z \in X \), where \( x \ll y \) is defined by \( 0 \in x \circ y \) and for every \( A, B \subseteq X \), \( A \ll B \) is defined by for all \( a \in A \), there exists \( b \in B \) such that \( a \ll b \). In such case "\( \ll \)" is called the hyper order in \( X \).

**Definition 2.3** [12]: Let \( A \) be a non-empty subset of a hyper BCI-algebra \( X \). Then \( A \) is said to be a hyper BCI-ideal of \( X \) if

1. \( 0 \in A \),
2. \( x \circ y \ll A \) and \( y \in A \) imply \( x \in A \) for all \( x, y \in X \).

**Definition 2.4**: For an intuitionistic fuzzy set \( A \) in \( X \) and \( t, s \in [0,1] \), the set

\[
A_{(t,s)} = \{x \in X / \mu_A(x) \geq t, \nu_A(x) \leq s\}
\]

is called a level subset of \( A \).

**Definition 2.5** [12]: Let \((H, \circ)\) be a hyper BCI-algebra. Then \((H, \circ, 0)\) is a BCI-algebra if and only if \( H = S_1 = \{x \in H / x \circ x = \{0\}\} \).

**Definition 2.6** [12]: Hypergroup is defined as a hyperstructure \((X, \cdot)\) such that the following axioms hold:

3. \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \) for all \( x, y, z \in X \),

4. \( x \cdot X = X \cdot x = X \) for all \( x \in X \).

Where \( x \cdot y = y \circ (0 \circ x) \) for all \( x, y \in X \).

**Theorem 2.7** [12]: Let \((X, \circ)\) be a hyper BCI-algebra and satisfy the following conditions:
(5) \( x \in a \circ (a \circ x) \),

(6) \( x \circ (0 \circ y) = y \circ (0 \circ x) \).

Then \((X, \cdot)\) is a hypergroup.

**Definition 2.8:** An intuitionistic fuzzy relation on any set \( X \) is an intuitionistic fuzzy set
\[
B = \langle \mu_B, \nu_B \rangle \text{ where } \mu_B : X \times X \rightarrow [0,1] \text{ and } \nu_B : X \times X \rightarrow [0,1].
\]

**Definition 2.9:** If \( B \) is an intuitionistic fuzzy relation on a set \( X \) and \( A \) is an intuitionistic fuzzy set in \( X \), then \( B \) is an intuitionistic fuzzy relation on \( A \) if
\[
\mu_B(x, y) \leq \min \{ \mu_A(x), \mu_A(y) \}
\]
and
\[
\nu_B(x, y) \geq \max \{ \nu_A(x), \nu_A(y) \}, \forall x, y \in X.
\]

**Definition 2.10:** If \( A \) is an intuitionistic fuzzy set in a set \( X \), the strongest intuitionistic fuzzy relation on \( X \) is an intuitionistic fuzzy relation of \( X \) is \( B_A = \langle (\mu_B)_{\mu_A}, (\nu_B)_{\nu_A} \rangle \), given by
\[
(\mu_B)_{\mu_A}(x, y) = \min \{ \mu_A(x), \mu_A(y) \}
\]
and
\[
(\nu_B)_{\nu_A}(x, y) = \max \{ \nu_A(x), \nu_A(y) \}, \forall x, y \in X.
\]

**Definition 2.11:** Let \( A \) \& \( B \) be intuitionistic fuzzy sets in a set \( X \). The Cartesian product of \( A \) and \( B \) is defined by
\[
(\mu_A \times \mu_B)(x, y) = \min \{ \mu_A(x), \mu_B(y) \}
\]
\[
(\nu_A \times \nu_B)(x, y) = \max \{ \nu_A(x), \nu_B(y) \} \text{ for all } x, y \in X.
\]

**Definition 2.12 [6]:** Let \( X \) and \( Y \) be hyper BCI-algebras. A mapping \( f : X \rightarrow Y \) is called a hyper homomorphism, if
\[
(7) \ f(0) = 0,
\]
\[
(8) \ f(x \circ y) = f(x) \circ f(y).
\]

### 3. Distributive Hyper BCI-Ideals

**Definition 3.1:** A non-empty set \( A \) of a hyper BCI-algebra \( X \) is called a distributive hyper BCI-ideal if it satisfies (1) and
\[
(9) \ ((x \circ z) \circ z) \circ (y \circ z) << A \text{ and } y \in A \Rightarrow x \in A.
\]

**Example 3.2:** Consider a hyper BCI-algebra \( X = \{0, a, b, c\} \) with the following Cayley
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\[ \begin{array}{cccc} 
\circ & 0 & a & b \\
0 & \{0,a\} & \{0,a\} & \{0,a\} \\
a & \{a\} & \{0,a\} & \{0,a\} \\
b & \{b\} & \{b\} & \{0,a,b\} \\
c & \{c\} & \{c\} & \{0,a,c\} \\
\end{array} \]

\{0,a\}, \{0,a,b\} are the only hyper BCI-ideals in \( X \) which are also distributive hyper BCI-ideals of \( X \).

**Example 3.3:** Consider a hyper BCI-algebra \( X = \{0,a,b,c\} \) with the following Cayley table.

\[ \begin{array}{cccc} 
\circ & 0 & a & b & c \\
0 & \{0\} & \{0\} & \{b\} & \{b\} \\
a & \{a\} & \{0,a\} & \{b\} & \{b\} \\
b & \{b\} & \{b\} & \{0\} & \{0\} \\
c & \{c\} & \{b,c\} & \{a\} & \{0,a\} \\
\end{array} \]

It can be easily checked that \( A = \{0,a\} \) is a hyper BCI-ideal but \( A \) is not a distributive hyper BCI-ideal of \( X \) because \( ((c \circ b) \circ b) \circ (0 \circ b) \ll A \) and \( 0 \in A \) implies \( c \in A \) which is a contradiction.

**4. Intuitionistic Fuzzy Hyper BCI-Ideals**

**Definition 4.1:** An intuitionistic fuzzy set \( A \) in a hyper BCI-algebra \( X \) is an intuitionistic fuzzy hyper BCI-ideal if

1. \( x \ll y \) implies \( \mu_A(y) \leq \mu_A(x) \) and \( \nu_A(y) \geq \nu_A(x) \),
2. \( \mu_A(x) \geq \min \{ \inf_{u \in (x \cdot y)} \mu_A(u), \mu_A(y) \} \),
3. \( \nu_A(x) \leq \max \{ \sup_{v \in (x \cdot y)} \nu_A(v), \nu_A(y) \} \), \( \forall x, y \in X \).

**Definition 4.2:** An intuitionistic fuzzy set \( A \) in \( X \) is called an intuitionistic fuzzy distributive hyper BCI-ideal if satisfies (10) and

1. \( \mu_A(x) \geq \min \{ \inf_{v \in (x \cdot z) \cdot (y \cdot z)} \mu_A(v), \mu_A(y) \} \),
2. \( \nu_A(x) \leq \max \{ \sup_{v \in (x \cdot z) \cdot (y \cdot z)} \nu_A(v), \nu_A(y) \} \), \( \forall x, y, z \in X \).

**Example 4.3:** Consider a hyper BCI-algebra \( X = \{0,a,b,c\} \) with the following Cayley table.
Define an intuitionistic fuzzy set $A$ in $X$ by

\[
\begin{align*}
\mu_A(a) &= \mu_A(b) = 0.2, \mu_A(0) = 0.4, \mu_A(0) = 0.6 \\
\nu_A(a) &= \nu_A(b) = 0.6, \nu_A(0) = 0.2, \mu_A(0) = 0.1.
\end{align*}
\]

It is routine to verify that $A$ is an intuitionistic fuzzy hyper BCI-ideal of $X$ but not an intuitionistic fuzzy distributive hyper BCI-ideal of $X$. Because

\[
\mu_A(x) \geq \min \left\{ \inf_{u \in \{(x+z)\cap (y+z)\}} \mu_A(u), \mu_A(y) \right\}
\]

and

\[
\nu_A(x) \leq \max \left\{ \sup_{u \in \{(x+z)\cap (y+z)\}} \nu_A(u), \nu_A(y) \right\}
\]

are not satisfied for $x = b, y = a, z = c$.

**Example 4.4:** Consider a hyper BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{b}</td>
<td>{b}</td>
</tr>
<tr>
<td>a</td>
<td>{a}</td>
<td>{0}</td>
<td>{b}</td>
<td>{b}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{b}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>c</td>
<td>{c}</td>
<td>{b}</td>
<td>{a}</td>
<td>{0,a}</td>
</tr>
</tbody>
</table>

Define an intuitionistic fuzzy set $A$ in $X$ by

\[
\begin{align*}
\mu_A(a) &= \mu_A(c) = \mu_A(0) = 0.2, \mu_A(0) = 0.4 \\
\nu_A(c) &= \nu_A(0) = 0.6, \nu_A(0) = 0.2 & \mu_A(0) = 0.2.
\end{align*}
\]

It is routine to verify that $A$ is an intuitionistic fuzzy hyper BCI-ideal as well as an intuitionistic fuzzy hyper BCI-ideal of $X$.

**Example 4.5:** Consider a hyper BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0,a}</td>
<td>{0,a}</td>
<td>{b}</td>
<td>{b}</td>
</tr>
<tr>
<td>a</td>
<td>{a}</td>
<td>{0,a}</td>
<td>{b}</td>
<td>{b}</td>
</tr>
<tr>
<td>b</td>
<td>{b}</td>
<td>{b}</td>
<td>{0,a}</td>
<td>{0,a}</td>
</tr>
<tr>
<td>c</td>
<td>{c}</td>
<td>{b,c}</td>
<td>{a}</td>
<td>{0,a}</td>
</tr>
</tbody>
</table>
Define an intuitionistic fuzzy set $A$ in $X$ by

\[ \mu_A(c) = 0.1, \mu_A(b) = 0.2, \mu_A(a) = 0.3, \mu_A(0) = 0.5 \text{ and} \]

\[ \nu_A(c) = 0.7, \nu_A(b) = 0.5, \nu_A(a) = 0.4 \text{ & } \mu_A(0) = 0.3. \]

It can be easily evaluated that $A$ is an intuitionistic fuzzy hyper BCI-ideal of $X$ but $A$ is not an intuitionistic fuzzy distributive hyper BCI-ideal of $X$. Since

\[ \mu_A(c) \geq \min \left\{ \inf_{u \in (c \circ b) \circ (a \circ b)} \mu_A(u), \mu_A(a) \right\} \]

and $\nu_A(c) \leq \max \left\{ \sup_{u \in (c \circ b) \circ (a \circ b)} \nu_A(u), \nu_A(a) \right\}$ are not satisfied.

**Proposition 4.6:** An intuitionistic fuzzy set $A$ is an intuitionistic fuzzy hyper BCI-ideal of a hyper BCI-algebra $X$ iff $A_{\{t,s\}}$ is a hyper BCI-ideal of $X$ whenever $A_{\{t,s\}} \neq \phi$ and $t, s \in [0, 1]$.

**Proof.** Let $A$ be an intuitionistic fuzzy hyper BCI-ideal of $X$ and $A_{\{t,s\}} \neq \phi$ for $t, s \in [0, 1]$.

Since $\mu_A(0) \geq \mu_A(x) \geq t$ and $\nu_A(0) \leq \nu_A(x) \leq s$ for some $x \in A_{\{t,s\}}$, therefore $0 \in A_{\{t,s\}}$.

Let $x, y \in X$ such that $x \circ y \ll A_{\{t,s\}}$ and $y \in A_{\{t,s\}}$ implies $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$.

For any $v \in x \circ y$ there exists $w \in A_{\{t,s\}}$ such that $v \ll w$ which implies that

\[ t \leq \mu_A(w) \leq \mu_A(v) \]

and $s \geq \nu_A(w) \geq \nu_A(v)$ then

\[ \mu_A(x) \geq \min \left\{ \inf_{x \circ y} \mu_A(v), \mu_A(y) \right\} \geq \{t, t\} \geq t \]

and

\[ \nu_A(x) \leq \max \left\{ \sup_{x \circ y} \nu_A(v), \nu_A(y) \right\} \leq \{s, s\} \leq s \]

implies that $x \in A_{\{t,s\}}$.

So $A_{\{t,s\}}$ is a hyper BCI-ideal.

Conversely, suppose $A_{\{t,s\}}$ is a hyper BCI-ideal of $X$. For any $x \in X$ setting $\mu_A(x) = t$ and $\nu_A(x) = s$ then $x \in A_{\{t,s\}}$.

Since $0 \in A_{\{t,s\}}$ which implies that $\mu_A(0) \geq t, \nu_A(0) \leq s$ so

\[ \mu_A(0) \geq \mu_A(x) \]

and $\nu_A(0) \leq \nu_A(x), \forall x \in X$.

For any $x, y \in X$, let

\[ t = \min \left\{ \inf_{w \circ x \circ y} \mu_A(w), \mu_A(y) \right\} \]

and $s = \max \left\{ \sup_{w \circ x \circ y} \nu_A(w), \nu_A(y) \right\}$. 

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Then for \( y \in A \) and \( u \in x \circ y \) we have
\[
\mu_A(u) \geq \left\{ \inf_{w \in x \circ y} \mu_A(w) \right\} \geq \sup_{w \in x \circ y} \mu_A(w) = t
\]
and \( v_A(u) \leq \left\{ \sup_{w \in x \circ y} v_A(w) \right\} \leq \inf_{w \in x \circ y} v_A(w) = s \)
which implies that \( u \in A \) so \( x \circ y \in A \). Therefore
\[
\mu_A(x) \geq t = \left\{ \inf_{w \in x \circ y} \mu_A(w), \mu_A(y) \right\}
\]
and \( v_A(x) \leq s = \left\{ \sup_{w \in x \circ y} v_A(w), v_A(y) \right\} \).

Hence \( A \) is an intuitionistic fuzzy hyper BCI-ideal of \( X \).

**Proposition 4.7:** Let \( A \) be an intuitionistic fuzzy hyper BCI-ideal of \( X \) then \( x \circ y \in A \) implies that \( \mu_A(x) \geq \min \{ \mu_A(z), \mu_A(y) \} \) and \( v_A(x) \leq \max \{ v_A(z), v_A(y) \} \).

**Proof:** Since \( A \) is an intuitionistic fuzzy hyper BCI-ideal then
\[
\mu_A(x) \geq \min \left\{ \inf_{w \in x \circ y} \mu_A(u), \mu_A(y) \right\} \geq \min \{ \mu_A(z), \mu_A(y) \}
\]
and \( v_A(x) \leq \max \left\{ \sup_{w \in x \circ y} v_A(u), v_A(y) \right\} \leq \max \{ v_A(z), v_A(y) \} \) because \( x \circ y \in A \) implies that
\[
\mu_A(z) \leq \mu_A(x \circ y)
\]
and \( v_A(z) \geq v_A(x \circ y) \).

**Theorem 4.8:** Every intuitionistic fuzzy distributive hyper BCI-ideal is an intuitionistic fuzzy hyper BCI-ideal.

**Proof:** Let \( A \) be an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \). Then
\[
\mu_A(x) \geq \min \left\{ \inf_{w \in (x \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\}
\]
\[
= \min \left\{ \inf_{w \in (x \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\}
\]
\[
= \min \{ \inf_{w \in (x \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \} \text{ for all } x, y \in X,
\]
and
\[
v_A(x) \leq \max \left\{ \sup_{w \in (x \circ y) \circ (y \circ z)} v_A(u), v_A(y) \right\}
\]
\[
= \max \{ \sup_{w \in (x \circ y) \circ (y \circ z)} v_A(u), v_A(y) \} \text{ for all } x, y \in X.
\]

Hence \( A \) is an intuitionistic fuzzy hyper BCI-ideal of \( X \). The converse of the Theorem 4.8 may not be true as seen in examples 4.3, 4.5.
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**Theorem 4.9:** An intuitionistic fuzzy set $A$ of a hyper BCI-algebra $X$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X$ iff for all $t, s \in [0, 1]$, $A_{(t,s)}$ is a distributive hyper BCI-ideal of $X$, whenever $A_{(t,s)} \neq \emptyset$.

**Proof:** Suppose $A$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X$ and $A_{(t,s)} \neq \emptyset$ for $t, s \in [0, 1]$ since $\mu_A(0) \geq \mu_A(x) \geq t$ and $\nu_A(0) \leq \nu_A(x) \leq s$ for some $x \in A_{(t,s)}$, we get $0 \in A_{(t,s)}$. If $((x \circ z) \circ y) \circ (y \circ z) \ll A_{(t,s)}$ and $y \in A_{(t,s)}$ then for any $u \in ((x \circ z) \circ y) \circ (y \circ z)$ there exists $v \in A_{(t,s)}$ with $\mu_A(v) \geq t$ and $\nu_A(v) \leq s$ such that $u \ll v$ which implies that $t \leq \mu_A(v) \leq \mu_A(u)$ and $s \geq \nu_A(v) \geq \nu_A(u)$. Since $A$ is an intuitionistic fuzzy distributive hyper BCI-ideal therefore for all $x, y, z \in X$, we have

$$\mu_A(x) \geq \min \left\{ \inf_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\} \geq \min \{t, t\} = t$$

and $\nu_A(x) \leq \max \left\{ \sup_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \nu_A(u), \nu_A(y) \right\} \leq \max \{s, s\} = s$ because $y \in A_{(t,s)}$ with $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$.

So $x \in A_{(t,s)}$. Hence $A_{(t,s)}$ is a distributive hyper BCI-ideal of $X$. Conversely suppose that for all $t, s \in [0, 1]$, $A_{(t,s)} (\neq \emptyset)$ is a distributive hyper BCI-ideal which implies that $A_{(t,s)}$ is a hyper BCI-ideal of $X$ and hence $A$ is a intuitionistic fuzzy hyper BCI-ideal of $X$. We have to prove that $A$ is a intuitionistic fuzzy distributive hyper BCI-ideal of $X$.

i.e. $A$ has to satisfy

$$\mu_A(x) \geq \min \left\{ \inf_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\}$$

and $\nu_A(x) \leq \max \left\{ \sup_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \nu_A(u), \nu_A(y) \right\}$. If not then

$$\mu_A(x) < \min \left\{ \inf_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\}$$

and $\nu_A(x) > \max \left\{ \sup_{u \in ((x \circ z) \circ y) \circ (y \circ z)} \nu_A(u), \nu_A(y) \right\}$.

Taking $t_0, s_0$ satisfying

$$\mu_A(x_0) < t_0 < \min \left\{ \inf_{u \in ((x_0 \circ z_0) \circ y_0) \circ (y_0 \circ z_0)} \mu_A(u_0), \mu_A(y_0) \right\}$$

and $\nu_A(x_0) > s_0 > \max \left\{ \sup_{u \in ((x_0 \circ z_0) \circ y_0) \circ (y_0 \circ z_0)} \nu_A(u_0), \nu_A(y_0) \right\}$, then $((x_0 \circ z_0) \circ y_0) \circ (y_0 \circ z_0) \ll A_{(t_0, s_0)}$ and $y \in A_{(t_0, s_0)}$ implies that $x_0 \notin A_{(t_0, s_0)}$ which is a contradiction. So $A$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X$.

**Theorem 4.10:** For any subset $A$ of $X$, let $A$ be an intuitionistic fuzzy set in $X$ defined by
\[ \mu_A(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} s & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases} \]

for all \( x \in X \) where \( t, s \in (0, 1) \). Then \( A \) is a distributive hyper BCI-ideal if and only if \( A \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \).

**Proof:** Since \( A \) is a distributive hyper BCI-ideal of \( X \) therefore if \(((x \circ z) \circ z) \circ (y \circ z) \ll A \) and \( y \in A \) then \( x \in A \). Hence

\[ \mu_A(((x \circ z) \circ z) \circ (y \circ z)) = t = \mu_A(y) = \mu_A(x) \]

and \( \nu_A(((x \circ z) \circ z) \circ (y \circ z)) = s = \nu_A(y) = \nu_A(x) \)

which implies that

\[ \mu_A(x) = \min \left\{ \inf_{u \in ((x \circ z) \circ z) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\} \]

and \( \nu_A(x) = \max \left\{ \sup_{u \in ((x \circ z) \circ z) \circ (y \circ z)} \nu_A(u), \nu_A(y) \right\} \)

If at least one of \(((x \circ z) \circ z) \circ (y \circ z)\) does not a hyper order in \( A \) and \( y \notin A \) then

\[ \mu_A(x) \geq \min \left\{ \inf_{u \in ((x \circ z) \circ z) \circ (y \circ z)} \mu_A(u), \mu_A(y) \right\} \]

and \( \nu_A(x) \leq \max \left\{ \sup_{u \in ((x \circ z) \circ z) \circ (y \circ z)} \nu_A(u), \nu_A(y) \right\} \)

Since \( 0 \in A \) implies that \( \mu_A(0) = t \geq \mu_A(x) \) and \( \nu_A(0) = s \leq \nu_A(x) \) for all \( x \in X \). So \( A \) is an intuitionistic fuzzy distributive hyper BCI-ideal. Converse follows from Theorem (4.9).

**Theorem 4.11:** Let \( A_1, A_2 \) be intuitionistic fuzzy distributive hyper BCI-ideals of \( X \). Then \( A_1 \times A_2 \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \times X \).

**Proof:**

Since

\[ (\mu_{A_1} \times \mu_{A_2})(x, y) = \min \left\{ \mu_{A_1}(x), \mu_{A_2}(y) \right\} \]

\[ (\nu_{A_1} \times \nu_{A_2})(x, y) = \max \left\{ \nu_{A_1}(x), \nu_{A_2}(y) \right\} \]

then

\[ (\mu_{A_1} \times \mu_{A_2})(0, 0) = \min \left\{ \mu_{A_1}(0), \mu_{A_2}(0) \right\} \geq \min \left\{ \mu_A(x), \mu_A(y) \right\} = (\mu_{A_1} \times \mu_{A_2})(x, y) \]

and

\[ (\nu_{A_1} \times \nu_{A_2})(0, 0) = \max \left\{ \nu_{A_1}(0), \nu_{A_2}(0) \right\} \leq \max \left\{ \nu_A(x), \nu_A(y) \right\} = (\nu_{A_1} \times \nu_{A_2})(x, y) \]

so
(\(\mu_A \times \mu_A\))(0,0) \geq (\(\mu_A \times \mu_A\))(x, y)

and \((\bigvee_A \times \bigvee_A)(0,0) \leq (\bigvee_A \times \bigvee_A)(x, y)\), for all \((x, y) \in X\).

Since
\[
(\mu_{A_1} \times \mu_{A_2})(x_1, y_1) \circ (x_2, y_2) = (\mu_{A_1} \times \mu_{A_2})(x_1 \circ x_2, y_1 \circ y_2)
\]
and
\[
(\bigvee_{A_1} \times \bigvee_{A_2})(x_1, y_1) \circ (x_2, y_2) = (\bigvee_{A_1} \times \bigvee_{A_2})(x_1 \circ x_2, y_1 \circ y_2)
\]

Consider
\[
(\mu_{A_1} \times \mu_{A_2})(x_1, y_1)
= \min \left\{ (\mu_{A_1}(x_1), \mu_{A_2}(y_1)) \right\}
\geq \min \left\{ \min \left\{ \inf_{t_1 \in ((x_1 \circ x_3) \circ x_3, x_2 \circ x_3)} \mu_{A_1}(t_1), \mu_{A_2}(x_2) \right\}, \min \left\{ \inf_{t_2 \in ((y_1 \circ y_3) \circ y_3, y_2 \circ y_3)} \mu_{A_1}(t_2), \mu_{A_2}(y_2) \right\} \right\}
= \min \left\{ \min \left\{ \inf_{t_1 \in ((x_1 \circ x_3) \circ x_3, x_2 \circ x_3)} \mu_{A_1}(t_1), \mu_{A_2}(x_2) \right\}, \min \left\{ \inf_{t_2 \in ((y_1 \circ y_3) \circ y_3, y_2 \circ y_3)} \mu_{A_1}(t_2), \mu_{A_2}(y_2) \right\} \right\}
\]
and
\[
(\bigvee_{A_1} \times \bigvee_{A_2})(x_1, y_1)
= \max \left\{ \bigvee_{A_1}(x_1), \bigvee_{A_2}(y_1) \right\}
\leq \max \left\{ \max \left\{ \sup_{t_1 \in ((x_1 \circ x_3) \circ x_3, x_2 \circ x_3)} \bigvee_{A_1}(t_1), \bigvee_{A_2}(x_2) \right\}, \max \left\{ \sup_{t_2 \in ((y_1 \circ y_3) \circ y_3, y_2 \circ y_3)} \bigvee_{A_1}(t_2), \bigvee_{A_2}(y_2) \right\} \right\}
= \max \left\{ \max \left\{ \sup_{t_1 \in ((x_1 \circ x_3) \circ x_3, x_2 \circ x_3)} \bigvee_{A_1}(t_1), \sup_{t_2 \in ((y_1 \circ y_3) \circ y_3, y_2 \circ y_3)} \bigvee_{A_1}(t_2) \right\}, \max \left\{ \sup_{t_1 \in ((x_1 \circ x_3) \circ x_3, x_2 \circ x_3)} \bigvee_{A_1}(t_1), \sup_{t_2 \in ((y_1 \circ y_3) \circ y_3, y_2 \circ y_3)} \bigvee_{A_1}(t_2) \right\} \right\}
\]
which implies that \(A_1 \times A_2\) is an intuitionistic fuzzy distributive hyper BCI-ideal of \(X\).
**Proposition 4.12:** Let $A$ be an intuitionistic fuzzy distributive hyper BCI-ideal of $X$. Then

$$\mu_A(x) \geq \inf_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \mu_A(u) \text{ and } \nu_A(x) \leq \sup_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \nu_A(u).$$

**Proof:** Since $A$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X$. Then

$$\mu_A(x) \geq \min \left\{ \inf_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \mu_A(u), \mu_A(y) \right\}$$

$$= \inf_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \mu_A(u) \text{ because } \mu_A(0) = \mu_A(t) \text{ for all } t \in X,$$

Also,

$$\nu_A(x) \leq \max \left\{ \sup_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \nu_A(u), \nu_A(0) \right\}$$

$$= \sup_{u \in ((x \circ z) \circ z) \circ (0 \circ z)} \nu_A(u) \text{ because } \nu_A(0) = \nu_A(t) \text{ for all } t \in X.$$

**Theorem 4.13:** Let $A$ be an intuitionistic fuzzy set in a hyper BCI-algebra $X$ and let $\lambda_A$ be the strongest intuitionistic fuzzy relation on $X$. Then $A$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X$ iff $\lambda_A$ is an intuitionistic fuzzy distributive hyper BCI-ideal of $X \times X$.

**Proof:** Let $A$ be an intuitionistic fuzzy distributive hyper BCI-ideal of $X$ then

$$\lambda_{\mu_A}(0,0) = \inf \left\{ \mu_A(0), \mu_A(0) \right\}$$

$$\geq \inf \left\{ \mu_A(0), \mu_A(0) \right\}$$

$$= \lambda_{\mu_A}(x_1, x_2) \text{ for all } (x_1, x_2) \in X \times X,$$

and

$$\lambda_{\nu_A}(0,0) = \sup \left\{ \nu_A(0), \nu_A(0) \right\}$$

$$\leq \sup \left\{ \nu_A(0), \nu_A(0) \right\}$$

$$= \lambda_{\nu_A}(x_1, x_2) \text{ for all } (x_1, x_2) \in X \times X.$$

Consider

$$\lambda_{\mu_A}(x_1, x_2) = \inf \left\{ \mu_A(x_1), \mu_A(x_2) \right\}$$

$$\geq \min \left\{ \inf_{u \in ((x_1 \circ z) \circ z) \circ (y_1 \circ z)} \mu_A(u), \mu_A(y_1) \right\} \inf \left\{ \inf_{v \in ((x_2 \circ z) \circ z) \circ (y_2 \circ z)} \mu_A(v), \mu_A(y_2) \right\}$$

$$= \min \left\{ \inf_{u \in ((x_1 \circ z) \circ z) \circ (y_1 \circ z)} \mu_A(u), \mu_A(y_1) \right\} \inf \left\{ \inf_{v \in ((x_2 \circ z) \circ z) \circ (y_2 \circ z)} \mu_A(v), \mu_A(y_2) \right\} \lambda_{\mu_A}(y_1, y_2)$$

$$= \min \left\{ \inf \left\{ \mu_A(u), \mu_A(v) \right\}, \lambda_{\mu_A}(y_1, y_2) \right\},$$

where

$$u \in ((x_1 \circ z_1) \circ z_1) \circ (y_1 \circ z_1), v \in ((x_2 \circ z_2) \circ z_2) \circ (y_2 \circ z_2)$$

$$= \min \left\{ \inf \left\{ \mu_A(u), \mu_A(v) \right\}, \lambda_{\mu_A}(y_1, y_2) \right\}.$$
where 
\[ u \in ((x_1 \circ z_1) \circ z_1) \circ (y_1 \circ z_1), v \in ((x_2 \circ z_2) \circ z_2) \circ (y_2 \circ z_2) \]
\[ = \min \left\{ \lambda_{\mu_A}(u,v), \lambda_{\nu_A}(y_1, y_2) \right\} \]
where \( (u,v) \in (((x_1,x_2) \circ (z_1,z_2)) \circ (y_1,y_2)) \circ ((z_1,z_2)) \)

Also,
\[ \lambda_{\mu_A}(x_1, x_2) = \max \left\{ \nu_A(x_1), \nu_A(x_2) \right\} \]
\[ \geq \max \left\{ \max \left[ \sup_{w \in ((x_1 \circ z_1) \circ z_1)} \nu_A(u), \nu_A(y_1) \right], \max \left[ \sup_{v \in ((x_2 \circ z_2) \circ z_2)} \nu_A(v), \nu_A(y_2) \right] \right\} \]
\[ = \max \left\{ \max \left[ \sup_{w \in ((x_1 \circ z_1) \circ z_1)} \nu_A(u), \nu_A(y_1) \right], \max \left[ \sup_{v \in ((x_2 \circ z_2) \circ z_2)} \nu_A(v), \nu_A(y_2) \right] \right\} \]
\[ = \max \left\{ \sup_{w \in ((x_1 \circ z_1) \circ z_1)} \nu_A(u), \nu_A(y_1), \nu_A(y_2) \right\}, \lambda_{\nu_A}(y_1, y_2) \]
where 
\[ u \in ((x_1 \circ z_1) \circ z_1) \circ (y_1 \circ z_1), v \in ((x_2 \circ z_2) \circ z_2) \circ (y_2 \circ z_2) \]
\[ = \max \left\{ \sup \nu_A(u, v), \nu_A(y_1, y_2) \right\} \]

where 
\[ u \in ((x_1 \circ z_1) \circ z_1) \circ (y_1 \circ z_1), v \in ((x_2 \circ z_2) \circ z_2) \circ (y_2 \circ z_2) \]
\[ = \max \left\{ \sup \lambda_{\mu_A}(u,v), \lambda_{\nu_A}(y_1, y_2) \right\} \]
\[ = \sup \lambda_{\mu_A}(u,v), \lambda_{\nu_A}(y_1, y_2) \]
where 
\[ (u,v) \in (((x_1,x_2) \circ (z_1,z_2)) \circ (y_1,y_2)) \circ ((z_1,z_2)) \circ ((y_1,y_2) \circ (z_1,z_2)) \]
which implies that \( \lambda_{\mu_A} \) and \( \lambda_{\nu_A} \), they are intuitionistic fuzzy distributive hyper BCI-ideals of \( X \times X \).

Conversely suppose that \( \lambda_{\mu_A} \) and \( \lambda_{\nu_A} \), they are intuitionistic fuzzy distributive hyper BCI-ideals of \( X \times X \) then for all \((x,y) \in X \times X\).

\[ \mu_A(0) = \min \{ \mu_A(0), \mu_A(0) \} = \lambda_{\mu_A}(x, y) = \min \{ \mu_A(x), \mu_A(x) \} = \mu_A(x) \]
and 
\[ \nu_A(0) = \max \{ \nu_A(0), \nu_A(0) \} = \lambda_{\nu_A}(x, y) = \max \{ \nu_A(x), \nu_A(x) \} = \nu_A(x) \]
which implies that \( \mu_A(0) \geq \mu_A(x) \) and \( \nu_A(0) \leq \nu_A(x) \) \( \forall x \in X \).

\[ \min \{ \mu_A(x_1), \mu_A(y_1) \} = \lambda_{\mu_A}(x_1, y_1) \]
\[ = \min \left\{ \inf_{w \in ((x_1 \circ y_1) \circ y_1) \circ y_1} \lambda_{\mu_A}(w), \lambda_{\mu_A}(x_2, y_2) \right\} \]

Also,
\[ \max \left\{ \nu_A(x_1), \nu_A(y_1) \right\} = \lambda_{\nu_A}(x_1, y_1) \]
\[ = \max \left\{ \sup_{w \in ((x_1 \circ y_1) \circ y_1) \circ y_1} \nu_A(w), \nu_A(x_2, y_2) \right\} \]
so if we put \( y_1 = y_2 = y_3 = 0 \) or \( (x_1 = x_2 = x_3 = 0) \) then we get
\[ \mu_A(x_1) \geq \min \left\{ \inf_{w \in ((x_1 \circ y_1) \circ y_1) \circ y_1} \mu_A(u), \mu_A(x_2) \right\} \]
and \( \nu_A(x_1) \leq \max \left\{ \sup_{w \in ((x_1 \circ y_1) \circ y_1) \circ y_1} \nu_A(u), \nu_A(x_2) \right\} \]
or \( \mu_A(y) \geq \min \left\{ \inf_{\forall \in (y_1, y_2, y_3), (y_2, y_3), y_3} \mu_A(v), \mu_A(y_3) \right\} \)

\( \nu_A(y) \leq \max \left\{ \sup_{\forall \in (y_1, y_2, y_3), (y_2, y_3), y_3} \nu_A(v), \nu_A(y_3) \right\} \)

which implies that \( A \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \).

**Theorem 4.14:** Let \( f : X \rightarrow Y \) be an onto hyper homomorphism from a hyper BCI-algebra \( X \) to a hyper BCI-algebra \( Y \). If \( A_2 \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( Y \) then BCI hyper homomorphic pre image \( A_1 \) of \( A_2 \) under \( f \) is also an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \).

**Proof:** Define \( A_2(f(x)) = A_1(x), \forall x \in X \) and since \( A_2 \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( Y \) with \( f(x) \in Y \).

Therefore, \( \mu_{A_1}(0) = \mu_{A_2}(f(0)) \geq \mu_{A_2}(f(x)) = \mu_{A_1}(x) \)

and \( \nu_{A_1}(0) = \nu_{A_2}(f(0)) \leq \nu_{A_2}(f(x)) = \nu_{A_1}(x) \forall x \in X. \)

\[ \mu_{A_1}(x) = \mu_{A_2}(f(x)) \]

\[ \geq \min \left\{ \inf_{u \in (f(x), y', z') \in (y', z')} \mu_{A_2}(u), \mu_{A_2}(y') \right\} \]

\[ = \min \left\{ \inf_{u \in (f(x), y, z) \in (y, z)} \mu_{A_2}(u), \mu_{A_2}(f(x)) \right\} \]

\[ = \min \left\{ \inf_{v \in (x, y, z) \in (y, z)} \mu_{A_1}(v), \mu_{A_1}(y) \right\} \text{because } f \text{ is onto.} \]

Also,

\[ \nu_{A_1}(x) = \nu_{A_2}(f(x)) \]

\[ \leq \max \left\{ \sup_{u \in (f(x), y', z') \in (y', z')} \nu_{A_2}(u), \nu_{A_2}(y') \right\} \]

\[ = \max \left\{ \sup_{u \in (f(x), y, z) \in (y, z)} \nu_{A_2}(u), \nu_{A_2}(f(x)) \right\} \]

\[ = \max \left\{ \sup_{v \in (x, y, z) \in (y, z)} \nu_{A_1}(v), \nu_{A_1}(y) \right\} \text{because } f \text{ is onto.} \]

Therefore \( A_1 \) is an intuitionistic fuzzy distributive hyper BCI-ideal of \( X \).

**References**


