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Metric Dimension of Composition Product of Graphs

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Abstract

A subset of vertices S resolves a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in S. The metric dimension of G is the minimum cardinality of a resolving set of G. For the graphs G1 = (V1;E1) and G2 = (V2;E2) its composition product is denoted by G1[G2] is the graph whose vertex set is V1 £ V2 and two vertices (u; v) and (x; y) are adjacent in G1[G2] whenever ux 2 E1, or, u = x and vy 2 E2. In this paper, we completely determined the metric dimension of the composition product of paths, paths and cycles, complete graphs, complete graphs and paths, path and stars.

Keywords: Metric Dimension, Metric basis, Resolving sets, Composition Product

AMS Subject Classification number: 05C20

Introduction

All the graphs considered in this paper are undirected, simple, finite and connected. We use standard terminology, the terms not defined here may found in [1, 5]. For each ordered subset $S = \{s_1, s_2, \ldots, s_k\}$ of V, each vertex $v \in V$ can be associated by a vector of distances denoted by $\Gamma(v/S) = (d_G(s_1, v), d_G(s_2, v), \ldots, d_G(s_k, v))$. The set S is said to be a resolving set of G, if $\Gamma(v) \neq \Gamma(u)$, for every $u, v \in V - S$. A resolving set of minimum cardinality is the *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The notion of metric dimension is introduced independently by F. Harary [4] and P.J.Slater [8, 9]. The cartesian product of graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G \Box H) = \{(a, v) : a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $a = b$ and $\{v, w\} \in E(H)$, or $v = w$ and $\{a, b\} \in E(G)$. In [3] José Cáceres et al, obtained bounds on the metric dimension of cartesian products through doubly resolving sets. In particular, they discussed on a family of (highly connected) graphs with bounded metric dimension for which the metric dimension of the cartesian product is unbounded. One of the results in [3] is the following;

Theorem 1.1 (José Cáceres, Marýa L. Puertas, Carmen Hernando, Mercè Mora, Ignacio M. Pelayo, Carlos Seara, David R. Wood [3]). *For all $n \geq 1$ and $m \geq 3$ we have,*

$$\beta(K_n \Box C_m) = \begin{cases} 2, & if\ n = 1 \\ 2, & if\ n = 2 \text{ and } m \text{ is odd} \\ 3, & if\ n = 2 \text{ and } m \text{ is even} \\ 3, & if\ n = 3 \\ 3, & if\ n = 4 \text{ and } m \text{ is even} \\ 4, & if\ n = 4 \text{ and } m \text{ is odd} \\ n-2, & if\ n \geq 5. \end{cases}$$

In the next sections, we prove similar type of results for the composition product of graphs similar to that of wheels and hexagonal cellular networks obtained in [10, 11].

We recall the following results for immediate reference, which we use in the next sections.

Theorem 1.2 (B.Sooryanarayana [12]). *A graph G with $\beta(G) = k$, cannot have a subgraph isomorphic to $K_{2^k+1} - (2^{k-1} - 1)e$.*

Remark 1.3. *In particular if the metric dimension of a graph is 2, then the above theorem tells that G should not contain a subgraph isomorphic to $K_5 - e$.*

Theorem 1.4 (S.Khuller, B.Raghavachari. A.Rosenfeld [6]). *The metric dimension of a graph G is 1 if and only if G is a path.*

Theorem 1.5 (F. Harary and R.A. Melter [4]). *The metric dimension of a cycle C_n is 2 for all $n \geq 3$.*

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Theorem 1.6 (F. Harary and R.A. Melter [4]). *The metric dimension of a non-trivial complete graph on n vertices is $n - 1$.*

Lemma 1.7. *No metric basis of a wheel on $n(\geq 4)$ vertices contains its central vertex*

Theorem 1.8 (B. Shanmukha, B. Sooryanarayana, K.S. Harinath [10]). *For given positive integer,*

$$\beta(W_{1,n}) \quad = \quad \begin{cases} 3, & if \ n \in \{3, 6\} \\ \lfloor \frac{2n+2}{5} \rfloor, & otherwise \end{cases}$$

Composition Product of Graphs

Definition 2.1. *The composition product of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph G, denoted by $G = G_1[G_2]$, whose vertex set is $V_1 \times V_2$ and two vertices (u_i, v_l) and (u_j, v_m) are adjacent in G whenever $\{u_i u_j \in E_1\}$ or $\{u_i = u_j$ and $v_l v_m \in E_2\}$.*

Remark 2.2. *Since the composition product of a complete graph K_m with a complete graph K_n is a complete graph K_{mn}, it easily follows from the Theorem 1.6 that,*

$$\beta(K_m[K_n]) = mn - 1, \text{ for all } m, n \geq 2.$$

Definition 2.3. *Let $G = G_1[G_2]$. Then for each i, $1 \leq i \leq m$, we define the horizontal projection H_i of G as*

$$H_i = \{(u_i, v_j) : u_i \in V(G_1), v_j \in V(G_2), 1 \leq j \leq |V(G_2)|\},$$

for each i, $1 \leq i \leq |V(G_1)|$

In next sections of this chapter we estimate the metric dimension for graphs obtained by taking the composition product of several combinations of graphs such as; the composition product of paths, paths and cycles, cycles and paths, complete graphs, complete graphs and paths, paths and stars. Some of these results are extensions of the earlier work of F. Harary and R.A. Melter [4].

Lower bounds for Metric Dimension of G[Pn]

In this section we determine lower bounds for the composition product of graphs. These bounds are not tight for the graphs of very small order. For such graphs the actual lower bounds are determined in the next sections while obtaining their upper bounds.

Observation 3.1. *The composition product $P_2[P_2] \cong K_4$ and hence it follows from the Theorem 1.6 that the metric dimension of $P_2[P_2]$ is 3.*

Observation 3.2. *Let $G = P_m[P_n]$ and for $m, n \geq 2$ then for any two vertices $u, v \in H_i$ and a vertex $x \notin H_i$, $d(x, u) = d(x, v)$. (This follows by noting the fact that each vertex in H_{i-1} and H_{i+1} is adjacent to every vertex in H_i)*

Lemma 3.3. *Let S be a metric basis for a graph G. Then the set $V - S$ can have at most one vertex equidistant from every element in S.*

Proof. Follows immediately by the definition of metric dimension. □

Lemma 3.4. *Let G be a non-trivial graph and S be a metric basis for the graph $G[P_n]$. Let $S_i = S \cap H_i$ for $1 \leq i \leq m$. Then $H_i - S$ has at most one vertex non-adjacent to any vertex in S_i, for every i, $1 \leq i \leq m$ and at most one vertex adjacent to every vertex in S_i.*

Proof. Suppose to contrary that $x = (u_i, v_a)$ and $y = (u_i, v_b)$ be two distinct vertices in $H_i - S$ are at equal distance from every vertex in S_i. Then, $d(x, w) = d(y, w)$, for all $w \in S_i$ and $d(x, w) = d(y, w) = d_{P_n}(u_i, u_j)$ for any $w = S_k$, $k \neq i$. Hence S will not resolve $G[P_n]$, a contradiction. □

Definition 3.5. *Let S be any subset of the vertex set V of a graph $G(V, E)$. Then for any vertex $x \in V$ we define k^{th} neighborhood of x in S, denoted by $kN_S(x)$, as $kN_S(x) = \{y \in S : d_G(x, y) = k\}$.*

Note: $1N_S(x)$ is simply written as $N_S(x)$ and call an S-neighborhood of x.

Lemma 3.6. *Let S be a subset of vertex set V of a non-trivial graph G and $u, v \in V - S$. Then S resolves G if and only if $kN_S(u) \neq kN_S(v)$, for some k.*

Proof. If $kN_S(u) = kN_S(v)$ for every k, then we observe under the hypothesis of the lemma, that the vector associated to u and v are identical and hence S cannot be a metric basis. Conversely, if $kN_S(u) \neq kN_S(v)$, for some k, then either $kN_S(u) \not\subset kN_S(v)$ or $kN_S(v) \not\subset kN_S(u)$. Without loss of generality, we take $kN_S(u) \not\subset kN_S(v)$ (since u and v are interchangeable). But then for the vertex $w \in kN_S(u) - (kN_S(u) \cap kN_S(v))$, we have $d(u, w) = k$ and $d(v, w) \neq k$. Hence, S resolves G. □

Remark 3.7. *The statement of the above lemma coincides to the definition of P. J. Slater [8]*

Lemma 3.8. *Let S be a resolving set of a graph G. Then for a positive integer p and a vertex $w \in S$, the vertices $u, v \in pN_{(V-S)}(w)$ implies that $kN_{(S-\{w\})}(u) \neq kN_{(S-\{w\})}(v)$, for some k.*

Proof. If $kN_{(S-\{w\})}(u) = kN_{(S-\{w\})}(v)$, for every k and $u, v \in pN_{(V-S)}(w)$ then it is straightforward to see that the vectors associated to u and v are identical, so S will not resolve G, a contradiction. □

Remark 3.9. *For the case $p = 1$, the above Lemma 3.8, yields a well known result of F. Harray that $\beta(K_m) = m - 1$.*

Remark 3.10. *If $G = P_2[P_3]$. Then $G - \{(u_1, v_1)\}$ is isomorphic to the graph $K_5 - e$. Hence by the Remark 1.3, it follows that $\beta(G) \geq 3$.*

Remark 3.11. *Consider the graph $G = G_1[P_3]$, where G_1 is a connected graph of order 3 (i.e. either P_3 or C_3). Let $x = (u_1, v_1)$, $y = (u_3, v_1)$. Then, as $kN_S(x) = kN_S(y)$ for every $S \subseteq V(G) - \{(u_1, v_1), (u_3, v_2)\}$ and $k = 1, 2$, at least one of the element of the set $\{(u_1 v_2), (u_3, v_2)\}$ is in any metric basis M of G. Due to the symmetry of the graph, without loss of generality, we take $(u_1, v_2) \in M$. Further for the vertex $u = (u_1, v_3)$, we have $kN_S(x) = kN_S(y)$ for every $S \subseteq V - \{u, x\}$, so either x or u should be in M, say $x \in M$. By Lemma 3.3, M should have at least one element from each H_i. Hence $|M| \geq 4$.*

Lemma 3.12. *For any integer $n \geq 1$, and a graph G.*

$$\beta(G[P_n]) \geq \begin{cases} |V(G)|, & if\ n = 2 \\ |V(G)| + 1, & if\ \ n = 3\ and\ m = 2,\ 3 \\ 2|V(G)| & if\ n = 4 \\ |V(G)|\lceil \frac{n}{2} - 1 \rceil, & Otherwise \end{cases} . \tag{1}$$

Proof. By the above Lemma 3.4, it follows that every metric dimension of $G[P_n]$ should contain at least one vertex if $n = 2$, and at least $\lceil \frac{n}{2} - 1 \rceil$ vertices of H_i if $n \geq 5$ for each i, $1 \leq i \leq m$. For $n = 3$ and $m = 3$, as the graph G is a path or a complete graph, the result follows by the Remark 3.11. Further, when $n = 3$ and $m = 2$, the result follows by Remark 3.10 (since $G \cong P_2$). When $n = 4$ if a metric basis S has at most one vertices from any H_i, then $H_i - S$ contains two vertices such that both of them are either adjacent to or non-adjacent to the vertex in $S \cap H_i$, Hence by Lemma 3.4 we get a contradiction, so $|S| \geq 2|V(G)|$. □

Metric Dimension and a Basis for *Pm*[*Pn*]

Theorem 4.1. *For the given positive integers $m, n \geq 2$,*

$$\beta(P_m[P_n]) = \begin{cases} 3, & if\ m = 2\ and\ n = 2 \\ m, & if\ m \geq 3\ and\ n = 2 \\ m+1, & if\ m \leq 3\ and\ n = 3 \\ 2m, & if\ m \geq 2\ and\ n = 4 \\ 5, & if\ m = 2\ and\ n = 6 \\ m\left\lceil \frac{n}{2} - 1 \right\rceil, & otherwise \end{cases}.$$

Proof. Let $G = P_m[P_n]$. Let $u_1, u_2, \ldots, u_m$ be the vertices of the path P_m such that u_i is adjacent to u_j if and only if $j = i+1$, $1 \leq i < j \leq n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n such that v_i is adjacent to v_j if and only if $j = i+1$, $1 \leq i < j \leq n$.

Case 1: $n = 2$ and $m = 2$

Result follows by the Observation 3.1

Case 2: $n = 2$ and $m \geq 3$

In this case, by Lemma 3.12 we have $\beta(G) \geq m$. Now we see that the set $S = \{(u_i, v_1) : 1 \leq i \leq m\}$ resolves G. In fact, let $u = (u_i, v_2)$ and $v = (u_j, v_2)$ be any two vertices of G. Without loss of generality we take $j > i$ (due to symmetry in the graph). Consider the vertex $w = (u_k, v_1)$ where

$$k = \begin{cases} i, & if\ j \neq i+1 \\ i-1, & if\ j = i+1\ and\ i > 2 \\ 3, & if\ j = 2\ and\ i = 1 \end{cases}$$

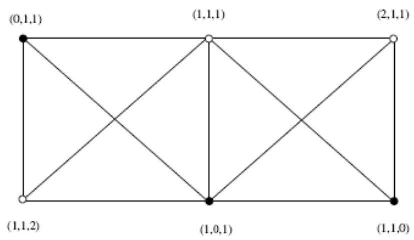

Figure 1: The graph $P_2[P_3]$.

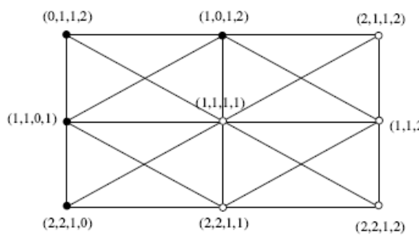

Figure 2: The graph $P_3[P_3]$.

The vertex w is adjacent to either u or v, but not both, Hence u and v lie at different distances from w. So, S resolves G. Thus, $\beta(G) \leq |S| = m$. Therefore, $\beta(G) = m$.

Case 3: $n = 3$ and $m = 2$

It follows by the Remark 1.3 that $\beta(G) \geq 3$. Let $\{(u_1, v_1), (u_2, v_2), (u_2, v_3)\}$. Then from the Figure 4 it follows that S resolves G. Hence $\beta(G) \leq 3$. Therefore, $\beta(G) = 3$.

Case 4: $n = 3$ and $m = 3$

In this case, by Lemma 3.12, we have $\beta(G) \geq 4$. Consider the set

$$S = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_1)\}.$$

From the Figure 2, it is clear that S resolves G. Hence $\beta(G) \leq |S| = 4$. Therefore $\beta(G) = 4$.

Case 5: $n = 3$ and $m \geq 4$

In this case, by Lemma 3.12 we have $\beta(G) \geq m$. Consider the set $S = \{(u_i, v_1) : 1 \leq i \leq m\}$. For the pair $u = (u_i, v_2)$, $v = (u_j, v_3)$, taking

$$w = \begin{cases} (u_i, v_1), & if\ j \neq i+1 \\ (u_{m-2}, v_1), & if\ j = m, i = m-1 \\ (u_{j+1}, v_1), & otherwise \end{cases}$$

and for the pair $u = (u_i, v_k)$, $v = (u_j, v_k)$, $k = 1, 2$, taking

$$w = \begin{cases} (u_{j+1}, v_1), & if\ j < m \\ (u_{i-1}, v_1), & if\ j = m \end{cases}$$

In either of the cases, we see that either $d(u, w) = 1$ or $d(v, w) = 1$ but not both. So, $d(u, w) \neq d(v, w)$. Other cases follows by symmetry. Hence S resolves G, so $\beta(G) \leq |S| = m$. Therefore $\beta(G) = m\lceil \frac{n}{2} - 1\rceil = m$.

Case 6: $n = 4$ and $m \geq 2$

In this case if a metric basis S contains at most one vertex from any H_i, then we get two vertices u, v such that either both of them are adjacent to the vertex $x \in S \cap H_i$ or none of them are adjacent to the vertex $x \in S \cap H_i$. In the first case $d(u, x) = d(v, x) = 1$ and in the second case $d(u, x) = d(v, x) = 2$. Further, as every vertex in H_j are equidistant from each vertex in H_i, for $i \neq j$, it follows that $kN_S(u) = kN_S(v)$, a contradiction by Lemma 3.8 to the fact that S is a metric basis. Thus, every metric basis should contain at least two element from each H_i. So, $\beta(G) \geq 2m$. Further, in the subset $S = \{(u_i, v_2), (u_i, v_3) : 1 \leq i \leq m\}$ of vertices of G, (i) for the pair of vertices

$x = (u_i, v_1), y = (u_j, v_4)$ with $j > i$ in $V - S$, the vertex $w = (u_j, v_3) \in S$ such that $d(y, w) = 1$ and $d(x, w) \neq 1$ if $j \neq i + 1$ or the vertex $w = (u_j, v_2) \in S$ such that $d(y, w) = 2$ and $d(x, w) = 1$ if $j = i + 1$ and (ii)for the pair of distinct vertices $x = (u_i, v_1), y = (u_j, v_1)$ in $V - S$ the vertex $w = (u_j, v_2) \in S$ if $j \neq i + 1$ (or $w = (u_j, v_3)$ if $j = i + 1$) such that $d(x, w) = 1$ and $d(y, w) \geq 2$ (or $d(x, w) = 2$ and $d(y, w) = 1$ in the later case). We note here that for the pair $x = (u_i, v_4), y = (u_j, v_4)$ in $V - S$ follows form (ii) by symmetry and the case $j < i$ also follows by symmetry by interchanging v_1 and v_4, v_2 and v_3. Thus, S resolves G. Hence $\beta(G) \leq 2m$. Therefore, by Lemma 3.12, we get $\beta(G) = 2m$.

Case 7: $n \geq 5$.

Let $S_1 = \{(u_1, v_{2k}) : 1 \leq k \leq \lceil \frac{n}{2} - 1\rceil\}$, $S_i = \{(u_i, v_{2k+1}) : 1 \leq k \leq \lceil \frac{n}{2} - 1\rceil\}$, for $2 \leq i \leq m$. Define a set S as follows;

$$S = \begin{cases} \{(u_i, v_j) : i = 1, 2\ and\ j = 2, 3\}, & if\ n = 5\ and\ m = 2 \\ \{(u_1, v_1)\} \cup \bigcup_{i=1}^{m} S_i, & if\ n = 6\ and\ m = 2 \\ \bigcup_{i=1}^{m} S_i, & otherwise \end{cases} \tag{2}$$

Claim: S resolves G

Since $d(x, y) = 1$ or 2, for all $x, y \in H_i$ and $d_G(x, z) = d_{P_n}(x, z) > 2$ for all $z \in H_j$ whenever $|i - j| > 2$, it suffices to observe the following:

1. $S \cap H_i$ resolves H_i, for every i, $1 \leq i \leq m$.
2. If $m = 2$, then there is no vertex in $H_1 - (S \cap H_1)$ that is adjacent to every vertex in $S \cap H_1$.
3. If $m = 3$, then there is no vertex in $H_1 - (S \cap H_1)$ at a distance 2 from every vertex in $S \cap H_1$.

The condition 1 is necessary because every vertex in $S-(S\cap H_i)$ are equidistance from the vertices of equal associated parities by the set $S\cap H_i$ (so S will not resolve these two vertices). Condition 2 and 3 are necessary because the vertex adjacent to every vertex in $S\cap H_1$ is also adjacent (or at a distance 2) to every vertex in $S\cap H_2$ (or $S\cap H_3$ in the second case) hence they are at equal distance from every vertex in S only when $n=2$ (or $n=3$ in the later case). However, if $m\geq 4$, then the S will resolve such vertices by a vertex in $S\cap H_4$. Hence the above conditions are also sufficient.

Now for each $x,y\in H_i-(S\cap H_i)$, there exists a vertex $w\in S\cap H_i$ such that w is adjacent to either x or y but not both (existence is certain because $n\geq 5$ and S chooses alternative vertices except the first vertex in each H_i). So $d(x,y)\neq d(y,w)$. Hence $S\cap H_i$ resolves H_i, hence the condition 1 holds, for all $m\geq 2$ and $n\geq 5$.

When $n\geq 6$, by Equation 2 we see that $S\cap H_1$ contains at least 3 vertices. Hence two vertices of H_1 are neither adjacent nor non-adjacent to every vertex in $S\cap H_1$ (since every vertex is adjacent to at least one vertex in H_1). Therefore conditions 2 and 3 hold.

When $n=5$ and $m=2$, we have by the choice of S in Equation 2 that no vertex in $H_1-(S\cap H_1)=\{(u_1,v_1),(u_1,v_4),(u_1,v_5)\}$ is adjacent to both vertex in $S\cap H_1=\{(u_1,v_2),(u_1,v_3)\}$. Hence the condition 2 holds.

When $n=5$ and $m=3$, we have by the choice of S in Equation 2 that each vertex in $H_1-(S\cap H_1)=\{(u_1,v_1),(u_2,v_2),(u_1,v_5)\}$ is adjacent to at least one vertex in $S\cap H_1=\{(u_1,v_2),(u_1,v_4)\}$. Hence the condition 3 holds.

When $n=5$ and $m\geq 4$, the conditions 2 and 3 are clear.

By the above claim and the Lemma 3.12, it follows that

$$\beta(G)=|S|=\begin{cases}5, & if\ m=2\ and\ n=6\\ m\lceil\frac{n}{2}-1\rceil, & otherwise\end{cases}$$

Hence the theorem. □

Metric Dimension and a Basis for Km[Pn]

Lemma 5.1. *A set S is a resolving set for the graph $G=K_m[P_n]$ if and only if the following hold;*

1. $S_i=S\cap H_i$ resolves H_i in G, for every i, $1\leq i\leq m$

2. For at most one i, $1\leq i\leq m$, there may be at most one vertex in H_i that is adjacent to every vertex in S_i.

Proof. If S resolves G, then, as the distance from every vertex in H_i is equidistance from each vertex in S_j for every $j\neq i$, it follows for each pair of vertices $u,v\in H_i$, there is a w in S_i such that $d(u,w)\neq d(v,w)$, so S_i resolves H_i. Hence the condition 1 holds. Further, if there are two distinct vertices $u\in H_i$ and $v\in H_j$ (i may be equal to j) such that u is adjacent to every vertex in $S\cap H_i$ and v is adjacent to every vertex in $S\cap H_j$, then, as these two vertices are adjacent to every vertices in $S\cap(H_i\cup H_j)$, it follows that the vectors associated to u and v by S are identical, a contradiction to the fact that S resolves G. Hence the condition 2 holds.

On the other hand, suppose that, for a subset S of G, both the conditions in the lemma are satisfied. Since the diameter of the graph G is 2, it follows that the vector associated to each vertex v of the graph by a set S is an element of Z_2^k, where $k=|S|$. If S will not resolves G, then there are two vertices $u\in H_i-S_i$ and $v\in H_j-S_j$ for some $1\leq i,j\leq m$ such that $d(u,w)=d(v,w)=1$ or 2 for every $w\in S$. Now, by condition 1, as S_i resolves H_i in G, we see that $v\notin H_i-S_i$, so $i\neq j$. But then $d(u,w)=d(v,w)=1$ (since $d(x,y)=2$ possible only if x and y are in H_i, for some i.) implies that u is adjacent to every vertex in S_i and v is adjacent to every vertex in S_j, which is a contradiction to condition 2 (since both H_i and H_j satisfies the condition and $i\neq j$). □

Theorem 5.2. *For the given positive integers $m, n \geq 2$,*

$$\beta(K_m[P_n]) = \begin{cases} mn-1, & if\ n=2 \\ 2m-1, & if\ n=3 \\ 2m, & if\ n=4,5 \\ 3m-1, & if\ n=6 \\ m\lceil \frac{n}{2}-1 \rceil, & if\ n \geq 7 \end{cases}$$

Proof. Let $G = K_m[P_n]$ and $u_1, u_2, \ldots, u_m$ be the vertices of the complete graph K_m. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n such that v_i is adjacent to v_j if and only if $j = i+1$, $1 \leq i < j \leq n$.

Case 1: $n = 2$ (also for $n = 1$ and $m > 1$)

In this case the graph is isomorphic to the complete graph and hence the result follows by the Theorem 1.6.

Case 2: $n = 3$

If a subset S has at most one vertex from each H_i, then $(V - S) \cap H_i$ contains a vertex adjacent to the vertex in $S \cap H_i$ for every i, $1 \leq i \leq m$. Hence, as $m \geq 2$, by Lemma 5.1, we see that S will not resolve G. Therefore S should have at least two elements from H_i, for each i except one, $1 \leq i \leq m$. Hence $\beta(G) \geq m + (m-1) = 2m-1$. To prove the reverse inequality, let $S_1 = \{(u_1, v_1)\}$ and $S_i = \{(u_i, v_1), (u_i, v_2)\}$ for $2 \leq i \leq m$. Then there is no vertex in H_i which is adjacent to every vertex in S_i for each i, $2 \leq i \leq m$. Hence by Lemma 5.1, we see that $S = \bigcup_{i=1}^{m} S_i$ resolves G, so $\beta(G) \leq |S| = 2m-1$. Thus, $\beta(G) = 2m-1$.

Case 3: $n = 4$ or $n = 5$

We first observe that if $|S_i|$=1, for any i, then either two vertices of H_i are adjacent to the vertex in S_i or two vertices are non-adjacent to the vertex in S_i. In the first case by condition 2, of Lemma 5.1, S will not resolve G and in the second case S_i will not resolve H_i in G, so by condition 1 of the Lemma 5.1, S will not resolve G. Hence every metric basis should contain at least 2 vertices in H_i, so $\beta(G) \geq 2m$. To prove the reverse inequality, let $S_i = \{(u_i, v_2), (u_i, v_3)\}$, for $i = 1, 2, \ldots, m$. Then for each i, $1 \leq i \leq m$, the vertices in $H_i - S_i$ is adjacent to at most one vertex in S_i (so condition 2 holds) and at most one vertex is non-adjacent any vertex in S_i, so S_i resolves H_i in G (i.e the condition 1 holds). Hence, by Lemma 5.1, $S = \bigcup_{i=1}^{m} S_i$ resolves G, Thus, $\beta(G) \leq 2m$. Therefore $\beta(G) = 2m$.

Case 4: $n = 6$

If there is a resolving set S having exactly two vertices of H_i, for any i, $1 \leq i \leq m$, then by condition 1 of Lemma 5.1, S_i should resolve H_i in G. But, the distance between any two non-adjacent vertices in G is 2, it follows that each assignment to the vertex in $H_i - S_i$ by the set S_i is a member of $A = \{(1,1), (1,2), (2,1), (22)\}$. So, as $|H_i - S_i| = 4$ and $|A| = 4$, we get a vertex v in H_i which receive a vector $(1,1)$. This implies that v is adjacent to every elements in S_i. Now, by the condition 2 of Lemma 5.1 this is possible only for one set i. Thus, every resolvable set of G should contain at least 3 vertices in H_i, for every i except one. Therefore, $\beta(G) \geq 3m-1$. To prove the reverse inequality, let $S_1 = \{(u_1, v_2), (u_i, v_4)\}$ and $S_i = \{(u_i, v_2), (u_i, v_3), (u_i, v_4)\}$. Then, as no vertex in H_i is adjacent to every vertex in S_i except of $i = 1$, and S_i resolves H_i in G (as at most one vertex is equidistant from every vertex in S_i), by Lemma 5.1 the set $S = \bigcup_{i=1}^{m} S_i$ resolves G. Hence $\beta(G) \leq |S| = 3m-1$. Therefore, $\beta(G) = 3m-1$.

Case 5: $n \geq 7$.

Let $S_i = \{(u_i, v_{2k+1}) : 1 \leq k \leq \lceil \frac{n}{2} - 1 \rceil\}$. Since $|S_i| \geq 3$, and $< H_i >$ is a path, it follows that no vertex in $H_i - S_i$ is adjacent to every vertex in S_i, so condition 1 of Lemma 5.1 is certain. Now by the choice of S_i it follows that for any two vertices u and v in $H_i - S_i$, either u or v is adjacent to a vertex $x \in S_i$, say $x = (u_1, v_j)$. Without loss of generality we take u is adjacent to x. Now choose the vertex $w = (u_1, v_{j+2})$ if $w \in S_i$ and is not adjacent to v, or else choose $w = (u_1, v_{j-2})$. Then $d(u, w) = 1$ and $d(v, w) = 2$, so S_i resolves G. Hence, by Lemma 5.1 we have $S = \bigcup_{i=1}^{m}$ resolves G. Thus, $\beta(G) \leq m\lceil \frac{n}{2} - 1 \rceil$. Therefore, by equation 3.12, it follows that $\beta(G) = m\lceil \frac{n}{2} - 1 \rceil$ for all $n \geq 7$. Hence the theorem. □

Metric Dimension and a Basis for Cm[Pn]

Theorem 6.1. *For the given integers m, n with $m \geq 3$ and $n \geq 2$,*

$$\beta(C_m[P_n]) = \begin{cases} 5, & if \;\; m = 3 \;\; and \;\; n = 2 \;\; or \; 3 \\ 6, & if \;\; m = 3 \;\; and \;\; n = 4 \;\; or \; 5 \\ 8, & if \;\; m = 3 \;\; and \;\; n = 6 \\ 4, & if \;\; m = 4 \;\; and \;\; n = 3 \\ 8, & if \;\; m = 4 \;\; and \;\; n = 4 \\ m, & if \;\; m \geq 5 \;\; and \;\; n = 2, 3 \\ 2m, & if \; m \geq 5 \; and \; n = 4 \\ m\left\lceil \frac{n}{2} - 1 \right\rceil, & otherwise \end{cases}$$

Proof. Let $G = C_m[P_n]$ and $u_1, u_2, \ldots, u_m$ be the vertices of the cycle C_m such that u_i adjacent to u_j if and only if either $|j - i| = 1$ or $n - 1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n such that v_i is adjacent to v_j if and only if $|j - i| = 1$.

Case 1: $m = 3$

The result follows by Theorem 5.2

Case 2: $m = 4$ and $n = 2k$ Let S be a resolving set for G and $S_i = S \cap H_i$, for $1 \leq i \leq m$. Now for any $x \in H_i$ the following hold;

1. $2N_S(x) = H_{i+2} \cup \{z \in H_i : xz \notin E(G)\}$ and
2. $1N_S(x) = H_{i-1} \cup H_{i+1} \cup \{z \in H_i : xz \in E(G)\}$

We first see that if there exist $x \in H_2 - S_2$ and $y \in H_4 - S_4$, then x is adjacent to a vertex in S_2 or y is adjacent to a vertex in S_4. Otherwise, $1N_S(x) = 1N_S(y) = S_3 \cup S_1$ and $2N_S(x) = 2N_S(y) = S_1 \cup S_3$ hence by Lemma 3.6, we have S will not resolve G, a contradiction. Therefore, by Lemma 3.12 $|S_2| > \lceil \frac{2k}{2} - 1 \rceil \Rightarrow |S_2| = k = \frac{n}{2}$ for all $n \geq 5$. Due to Horizontal symmetry of the graph, similar argument holds for each H_i, $1 \leq i \leq 4$. Thus, $|S| \geq 4\frac{n}{2} = 2n$. Further, for $n = 2$ or 4, we can avoid such vertices by taking respectively 1 or 2 elements from H_2 in S_2. . Therefore, in view of Lemma 3.12, we get

$$\beta(G) \geq \begin{cases} 4, & if \; k = 1 \\ 8, & if \; k = 2 \\ 2n, & if \; k \geq 3 \end{cases}$$

To prove the reverse inequality, we now consider the set

$$S = \left\{(u_i, v_{2p}) : 1 \leq i \leq m, 1 \leq p \leq \tfrac{n}{2}\right\}.$$

Let $u, v \in V - S$. If $u, v \in H_2$ and distinct (possible only if $k \geq 2$), then by the choice of S, there is a vertex in S_2 adjacent to u and not adjacent to v, so $1N_S(u) \neq 1N_S(v)$. Or if $u \in H_2$ and $v \in H_3$ (similarly in H_1), then $2N_S(u) = H_4 \cup \{z \in H_2 : xz \notin E(G)\} \neq H_1 \cup \{z \in H_3 : xz \notin E(G)\} = 2N_S(v)$. Lastly, if $u \in H_2$ and $v \in H_3$, then as the vertex adjacent to u is lies in $2N_S(v)$ but not in $2N_S(u)$, it follows that $2N_S(u) \neq 2N_S(v)$. The similar argument holds for all H_i by replacing H_2 as H_i due to symmetry. Hence, we conclude by Lemma 3.6, that S resolves G. Therefore, $\beta(G) \leq |S| = 2n$. Thus, $\beta(G) = 2n = 4k$.

Similar argument holds for the case $m = 4$ and $n = 2k + 1$. The set S taken in the above Case (ii) also serves as a resolving set for G in this case. Hence $\beta(G) = 4k = 4(n-1)/2 = 2n - 2$.

Case 3: $m \geq 5$ and $n = 2, 3$

Let $S = \{(u_i, v_1) : 1 \leq i \leq m\}$. Since, $m \geq 5$ we see that $1N_S(x) \neq 1N_S(y)$ whenever $x \in H_i - S$, $y \in H_j - S$ and $i \neq j$. Further, when $i = j$ (possible only if $n = 3$), the vertex (u_i, v_1) is adjacent to exactly one of x, y, so $1N_S(x) \neq 1N_S(y)$. Hence by Lemma 3.6, S resolves G. Therefore, $\beta(G) \leq |S| = m$. So, in view of Lemma 3.12, we conclude $\beta(G) = m$.

Case 4: $m \geq 5$ and $n = 4$

Let $S = \{(u_i, v_2), (u_i, v_3) : 1 \leq i \leq m\}$. Then, similar to above case wee see that $1N_S(x) \neq 1N_S(y)$ for any $x \in H_i - S$, $y \in H_j - S$, for every $1 \leq i, j \leq m$. By Lemma 3.6, $\beta(G) \leq |S| = 2m$. So, in view of Lemma 3.12, we conclude $\beta(G) = 2m$.

Case 5: $m \geq 4$ and $n \geq 5$

By Lemma 3.6, $\beta(G) \geq m\lceil \frac{n}{2} - 1 \rceil$.

Let $S = \{(u_i, v_3), (u_i, v_5), \ldots (u_i, v_{2\lceil \frac{n}{2}-1\rceil+1}) : 1 \leq i \leq m\}$.

Claim: S resolves G.

Since $m \geq 5$, for each $x \in H_i$ and $y \in H_j$, by the definition of composition product, there exists an index k such that $xz \in V(G)$ and $yz \notin V(G)$, for all $z \in H_k$. Therefore, it suffices to prove that, $S_i = S \cap H_i$ resolves H_i. Let u and v be any two vertices in $H_i - S$. Then, by the choice of S and $n \geq 5$, we can find a vertex w in $S \cap H_i$ such that w adjacent to u or v, but not both, and hence $d(u, w) \neq d(v, w)$. Hence the claim.

Therefore $\beta(G) = |S| = m\lceil \frac{n}{2} - 1 \rceil$. This completes the proof in all cases. □

Metric Dimension of Pn[G]

In this section we completely determine metric dimensions Pn[G], for every graph G of diameter at most two.

Theorem 7.1. *Let G be a non-trivial graph of diameter 2 and for every metric basis S of G there be a vertex in $V - S$ which is at a distance k from every vertex in S. Then $\beta(P_m[G]) = m\beta(G) + 1$ if and only if $1 \leq k < m \leq 3$. Otherwise (if such an vertex exists or not), $\beta(P_m[G]) = m\beta(G)$*

Proof. Since $d((u_i, v_j), (u_k, v_l)) = |j - i - 1|$ for every i, k, $1 \leq i, j \leq m$ and $1 \leq j, l \leq |V(G)|$, it follows for any metric basis M of $G[P_n]$ that the codes of two vertices in $(v - M) \cap H_i$ should differ by the metric basis S of G. Hence $M \supseteq \bigcup_{i=1}^{m} S_i$, where $S_i = \{(u_i, v_j) : v_j \in S\}$. Further, if there is a vertex, say x, in G at a distance k from every vertex in S for any metric basis S, then the co-ordinates of the codes generated by the set $S_1 \cup S_{k+1}$ for the vertices $y = (u_i, x)$ and $z = (u_{k+1}, x)$ are equal. Thus, for a valid code at least one vertex u should be in M such that $d(x, u) \neq d(y, u)$. Such a vertex u exists if and only if $k \geq 3$ (so $m \geq 4$). Now it is easy to observe that the set

$M = \bigcup_{i=1}^{m} S_i \cup \{(u_1, x)\}$ for $k \leq 2$ and $M = \bigcup_{i=1}^{m} S_i$ for $k \geq 3$ generates a valid code for $P_m[G]$. Hence the theorem. □

Corollary 7.2. *For the given positive integers $m, n \geq 1$,*

$$\beta(P_m[K_{1,n}]) = \begin{cases} 1, & if\ m = 1\ and\ n = 1,2 \\ n-1, & if\ m = 1\ and\ n \geq 3 \\ 3, & if\ m = 2\ and\ n = 1 \\ m, & if\ m \geq 3\ and\ n = 1 \\ m(n-1)+1, & if\ m = 2,3\ and\ n \geq 2 \\ m(n-1), & if\ m \geq 4\ and\ n \geq 2 \end{cases}$$

Proof. For $n = 1, 2$ result follows by Theorem 4.1 and Theorem 1.4. For $n \geq 3$, by the result of S. Kuller et al [6] on trees, the metric dimension of $K_{1,n}$ is $n-1$ and every metric basis should contain all the pendent vertices except the one, hence it follows that for every metric basis S of $G = K_{1,n}$, the central vertex is at a distance $k = 1$ from every vertex in S and the pendent vertex not in S is at a distance $k = 2$ from every vertex in S. Therefore, the result follows by the above Theorem 7.1 □

Corollary 7.3. *For given positive integers $m, n \geq 2$,*

$$\beta(P_m[K_n]) = \begin{cases} 2n-1, & if\ m = 2 \\ m(n-1), & if\ m \geq 3 \end{cases}$$

Proof. Since every metric basis of K_n contains $n-1$ vertices of K_n and the remaining vertex is adjacent to each vertex in the metric basis, the result follows immediately form the above Theorem 7.1. □

Corollary 7.4. *For given positive integers $m, n \geq 2$,*

$$\beta(P_m[W_{1,n}]) = \begin{cases} 1, & if\ m = n = 1 \\ 3, & if\ m = 2\ and\ n = 1 \\ m, & if\ m \geq 3\ and\ n = 1 \\ 2, & if\ m = 1\ and\ n = 2 \\ 2n-1, & if\ m = 2\ and\ n = 2,3 \\ m(n-1), & if\ m \geq 3\ and\ n = 2,3 \\ m\lfloor \frac{2n+2}{5} \rfloor + 1, & if\ m = 2,3\ and\ n = 4,5\ or\ n \geq 7 \\ m\lfloor \frac{2n+2}{5} \rfloor, & if\ m \geq 4\ and\ n = 4,5\ or\ n \geq 7 \\ 3m+1, & if\ m = 2,3\ and\ n = 6 \\ 3m, & if\ m \geq 4\ and\ n = 6 \end{cases}$$

Proof. By Lemma 1.7, it follows for all $n \geq 4$ that the central vertex is always at a distance 1 from each of the basis elements, by Theorem 1.8, we see that, there exists a rim vertex that is at a distance 2 from each of the basis elements (since every basis element not contains at least one rim vertex). Now, when $n = 1$ and $m = 1$, the result follows by Theorem 1.4. When $n = 1$ and $m \geq 2$ result follows by Theorem 4.1. For $n = 2, 3$ and $m = 1$, result follows by Theorem 1.5 whereas the case $m \geq 2$ follows by the Corollary 7.3. Finally, the case $n \geq 4$ follows by Theorem 1.8 and Theorem 7.1. □

Metric Dimension and a Basis for Pm[Cn]

Let $u_1, u_2, \ldots, u_m$ be the vertices of the path P_m such that u_i is adjacent to u_j if and only if $j = i+1$, $1 \leq i < n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path C_n such that v_i is adjacent to v_j if and only if $j = i+1$ for $1 \leq i < n$ and v_1 is adjacent to v_n.

Lemma 8.1. *Let S be any subset of the vertices of the graph $G[P_n]$ and $S_i = S \cap H_i$, where G is a graph of order m. Then S resolves G if and only if*

1. *S_i resolves H_i in G, for each i*

2. *If $m = 2$, then for at least one i, no vertex in $H_i - S_i$ is adjacent to every vertex in S_i*

3. *If $m = 3$, then either in H_1 or in H_3 every vertex is adjacent to at least one vertex in S_1 or S_3.*

Proof. Since $d(x, z) = d(y, z)$ for all $x, y \in H_i$ and $z \notin H_i$, condition 1 is certain. When $m = 2$, the graph is isomorphic to a complete graph, so no comments. Lastly $x \in H_1$ and $y \in H_3$ are not adjacent to any vertex respectively in S_1 and S_3, then they are at equal distance from each vertex in S only when $m = 3$, so S will not resolve G if $m = 3$. □

Theorem 8.2. *For the given integers $m \geq 1$, $n \geq 3$,*

$$\beta(P_m[C_n]) = \begin{cases} 2, & if \;\; m = 1 \\ 5, & if \; m = 2 \; and \; n = 3, 6 \\ 2m, & if \;\; m \geq 2 \; and \; n = 4, 5 \\ 2m, & if \;\; m \geq 3 \; and \; n = 3 \\ m \left\lceil \frac{n}{2} - 1 \right\rceil, & otherwise \end{cases}$$

Proof. Let $G = P_m[C_n]$ and for each subset S of $V(G)$, $S_i = S \cap H_i$.

Case 1: $m = 1$

In this case $G \cong C_n$ and hence by the Theorem 1.5, we get $\beta(G) = 2$.

Case 2: $m = 2$ and $n = 3$

The graph $G = P_2[C_3] \cong K_6$ and hence by the Theorem 1.6, we get $\beta(G) = 5$.

Case 3: $m = 2, 3$ and $n = 6$

If a set S contains at most 4 vertices, then it has exactly two elements from each of the sets S_1 and S_2 (otherwise S_i will not resolve H_i, because it is not a path). But then in each of the sets $H_i - S_i$ we see the codes of the 4 vertices in $H_i - S_i$ to be from the set $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Hence the condition 2 for the case $m = 2$ (condition 3 for the case $m = 3$)of Lemma 8.1 fails, so S will not resolve G. Thus, $\beta(G) \geq 5$. To prove the reverse inequality, consider the set

$$S = \{(u_i, v_j) : 1 \leq i \leq 2, j = 3, 5\} \bigcup \{(u_1, v_1)\}$$

. It is easy to verify that S resolves G, so $\beta(G) \leq 5$. Hence $\beta(G) = 5$.

Case 4: $m \geq 2$ and $n = 4$

Let $S = \{(u_i, v_j) | 1 \leq i \leq m, j = 1, 2\}$. Then each vertex in $H_i - S_i$ is adjacent to a exactly one vertex in S_i, for each i, $1 \leq i \leq m$, and S_i resolves H_i. Hence by Lemma 8.1, S resolves G. Therefore, in view of Lemma 3.12, we conclude $\beta(G) = |S| = 2m$.

Case 5: $m \geq 2$ and $n = 5$

Let $S = \{(u_1, v_2), (u_1, v_4)\} \cup \{(u_i, v_j) : \; 2 \leq i \leq m, j = 1, 2\}$. Then each vertex in $H_i - S_i$ is adjacent to a exactly one vertex in S_i, for each i, $2 \leq i \leq m$, and S_i resolves H_i for each i, $1 \leq i \leq m$. Hence by Lemma 8.1, S resolves G. Therefore, in view of Lemma 3.12, we conclude $\beta(G) = |S| = 2m$.

Case 6: $m \geq 2$ and $n = 2k + 5$, $k \in Z^+$

Let $S_i = \{(u_i, v_{2j+1}) : 1 \leq j \leq k+2\}$. Then, for each i, $1 \leq i \leq m$, every vertex in $H_i - S_i$ is adjacent to at least one vertex in S_i, no vertex is adjacent to every vertex in S_i (since $|S| \geq 3$ as $k \geq 1$) and for each pair of vertices in $H_i - S_i$ there is a vertex adjacent to one of them and not adjacent to the other, hence S_i resolves H_i. So, by Lemma 8.1, $S = \bigcup_{i=1}^{m} S_i$ resolves G. Therefore, in view of Lemma 3.12, we conclude $\beta(G) = |S| = m\lceil \frac{n}{2} - 1\rceil$.

Case 7: $m = 2$ and $n = 2k + 6$

Let $S_i = \{(u_i, v_{2j+1}) : 1 \leq j \leq k+2\}$. Then, for each i, $1 \leq i \leq m$, no vertex is adjacent to every vertex in S_i (since $|S| \geq 3$ as $k \geq 1$), at most one vertex in $H_i - S_i$ that is non-adjacent any vertex in S_i and for each pair of vertices in $H_i - S_i$ there is a vertex adjacent to one of them and not adjacent to the other, hence S_i resolves H_i. So, by Lemma 8.1, $S = \bigcup_{i=1}^{m} S_i$ resolves G. Therefore, in view of Lemma 3.12, we conclude $\beta(G) = |S| = m\lceil \frac{n}{2} - 1\rceil$.

Case 8: $m = 3$ and $n = 8$

If each S_i contains at most $\lceil \frac{n}{2} - 1\rceil = 3$ vertices and satisfies the condition 2 and 3 of Lemma 8.1, then we see that S_i contains a pair of vertices x and y such that both x and y are adjacent to exactly one vertex in S_i and non-adjacent to remaining vertices in S_i. Thus, $kN_S(x) = kN_S(y)$ for every k. Hence by Lemma 3.6, S will not resolve G. Thus, $\beta(G) \geq 10$. On the other hand from the Figure 3 we have $\beta(G) \leq 10$. Thus, we conclude that $\beta(G) = 10$.

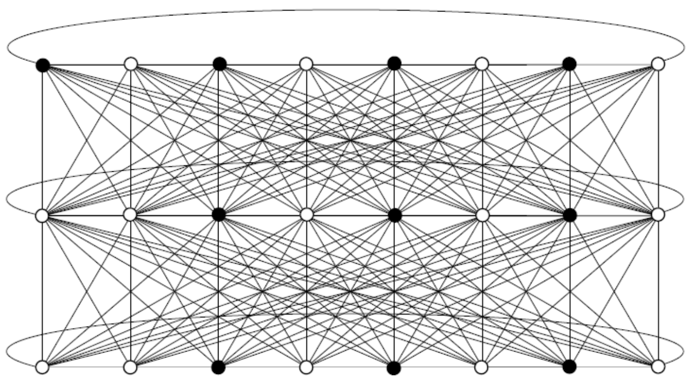

Figure 3: Metric basis (darkened vertices) of the graph P3[C8].

Case 9: $m \geq 3$ and $n \geq 10$ (or $m \geq 4$ and $n = 4$)

Let $S_i = \{(u_i, v_1), (u_i, v_4), (u_i, v_6), (u_i, v_9)\} \bigcup \{(u_i, v_{2k+9}) : 1 \leq k \leq (\lceil \frac{n}{2} - 1\rceil - 4)\}$. Since every vertex is adjacent to at least one vertex in S_i and no two vertices are adjacent to every vertex in S_i, S will resolve H_i. Thus, by Lemma 8.1, $S = \bigcup_{i=1}^{m} S_i$ resolves G for every $m \geq 3$, so $\beta(G) = |S| = m\lceil \frac{n}{2} - 1\rceil$. Therefore, in view of Lemma 3.12, we conclude $\beta(G) = m\lceil \frac{n}{2} - 1\rceil$. □

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