International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 3, Number 4 (2011), pp. 411-420

http://www.irphouse.com

Generalized k-normal Matrices

[1]S. Krishnamoorthy and [2]R. Subash

[1]Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Tamilnadu, India 612 001.
[2]Department of Mathematics, A.V.C. College of Engineering, Mannampandal, Mayiladuthurai, Tamilnadu, India 609305.
E-mail: subash_ru@rediffmail.com

Abstract

A matrix $A \in \mathbb{C}_{n\times n}$ is called generalized k-normal provided that there is a positive definite k-hermite matrix H such that HAH is k-normal. In this paper, these matrices are investigated and their canonical form, invariants and relative properties in the sense of congruence are obtained.

AMS classification: 15A04, 15A18.

Keywords: Generalized k-normal matrix, congruence, canonical form and invariant.

Introduction

Let $\mathbb{C}_{n\times n}$ be the space of nxn complex matrices. Let 'k' be a fixed product of disjoint transposition in S_n={1,2,…n} (hence, involutary) and let K be the associated permutation matrix of k. Two matrices $A, B \in \mathbb{C}_{n\times n}$ are called k-unitarly similar provided that there exists a k-unitarly matrix U such that $B = KU^*AU$, where U^* is the conjugate transpose matrix of U. For a matrix $A \in \mathbb{C}_{n\times n}$, there exists a k-unitary matrix U such that KU^*AU is an upper triangular matrix and A is k-unitarly similar to a k-diagonal matrix if and only if A is k-normal. Two matrices $A, B \in \mathbb{C}_{n\times n}$ are congruent if there exists a nonsingular matrix $P \in \mathbb{C}_{n\times n}$ such that $B = KP^*AP$. It is easy to see that, for a given matrix $A \in \mathbb{C}_{n\times n}$, there exists a nonsingular matrix

$P \in \mathbb{C}_{n\times n}$ such that $K P^* AP$ is an upper triangle matrix since k-unitary similar matrices A, B are congruent.

In this paper, we obtain the canonical form and invariants for generalized k-normal matrices in the sense of congruence. We give some determinantal inequalities for generalized k-normal matrices and their k-hermite part and skew k-hermite part.

Basic Definitions

Definition 2.1: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be k-hermitian, if $A = K A^* K$.

That is, $a_{ij} = \overline{a}_{k(j)\,k(i)}$, i,j=1,2,…,n.

Example 2.2: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ -i & 0 & 0 \end{bmatrix}$ is k-hermitian.

Definition 2.3: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be skew k-hermitian, if $A = -K A^* K$.

That is, $a_{ij} = -\overline{a}_{k(j)\,k(i)}$, i,j=1,2,…,n .

Example 2.4: $A = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}$ is skew k-hermitian.

Generalized k-normal matrices

Definition 3.1[4]: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be k-normal, if $AA^*K = K A^* A$.

That is, $a_{ij}\, \overline{a}_{n-k(j)+1\; k(i)} = \overline{a}_{k(j)\; n-k(i)+1}\, a_{ij}$; i, j = 1, 2… n.

Example 3.2: $A = \begin{bmatrix} 0 & -i & -3i \\ -i & -3i & 0 \\ -3i & 0 & -i \end{bmatrix}$ is k-normal.

Definition 3.3[4]: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be k-unitary, if $AA^*K = K A^* A = K$.

Example 3.4: $A = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$ is k-unitary.

Definition 3.5: A matrix $A \in \mathbb{C}_{n\times n}$ is called generalized k-normal if there is a positive definite k-hermite matrix H such that HAH is k-normal.

Example 3.6: $H = \begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix}$ is a positive definite k-hermite matrix.

Then $A = \begin{bmatrix} -i & 1 \\ 0 & i \end{bmatrix}$ is generalized k-normal.

Theorem 3.7: $A \in \mathbb{C}_{n\times n}$ is generalized k-normal if and only if there exists a positive definite k-hermite matrix X such that $K\,A^* X\,A = A\,X\,A^* K$.

Proof: Assume that A is generalized k-normal matrix. Then by definition (3.5), there exists a positive definite k-hermite matrix H such that HAH is k-normal.

Therefore,

$$K(H\,AH)^*(H\,AH) = (H\,AH)(HAH)^* K$$

$$\Rightarrow\; K\,H^* A^* H^* H\,AH = H\,AH\,H^* A^* H^* K$$

$$\Rightarrow\; K(K\,H\,K)A^*(K\,H\,K)\,H\,AH = H\,AH(K\,H\,K)A^*(K\,H\,K)K$$

$$\Rightarrow\; H\,K\,A^*(K\,H)^2 AH = H\,A(H\,K)^2 A^* K\,H$$

$$\Rightarrow\; K\,A^*(K\,H)^2 A = A(H\,K)^2 A^* K \tag{1}$$

Since H is a positive definite, H is non-singular. Then H^2 is positive definite. Let $X = H^2 = H^2K^2 = (H\,K\,)^2 = (K\,H\,)^2$ .Then X is a positive definite k-hermite matrix. There fore (1) $\Rightarrow\; K\,A^* X\,A = A\,X\,A^* K$.

Conversely, Assume that $K\,A^* X\,A = A\,X\,A^* K$ and X is a positive definite k-hermite matrix. Then there exists a positive definite k-hermite matrix H such that $X = H^2 = H^2K^2 = (H\,K\,)^2 = (K\,H\,)^2$.

Thus

$$K\,A^*(K\,H)^2 A = A(H\,K)^2 A^* K$$

$$\Rightarrow\; H\,K\,A^* K\,H\,K\,H\,AH = H\,AH\,K\,H\,K\,A^* K\,H$$

$$\Rightarrow\; (K\,H^* K)K\,A^* H^* H\,AH = H\,AH\,H^* A^* K(K\,H^* K)$$

$$\Rightarrow\; K\,H^* A^* H^* H\,AH = H\,AH\,H^* A^* H^* K$$

$$\Rightarrow\; K(H\,AH)^*(H\,AH) = (H\,AH)(H\,AH)^* K$$

Hence HAH is k-normal and then A is generalized k-normal.

Theorem 3.8: $A \in \mathbb{C}_{n\times n}$ is generalized k-normal if and only if A is congruent to a k-diagonal matrix.

Proof: Assume that A is generalized k-normal matrix. Then by definition (3.5), there exists a positive definite k-hermite matrix H such that HAH is k-normal. By Schur's theorem [2], there exists a k-unitary matrix U such that $KU^*(HAH)U$ is k-diagonal matrix. Let $P = HU \Rightarrow P^* = U^*H^* = U^*KHK \Rightarrow P^*H = U^*KHKH \Rightarrow P^*H = U^*(KH)^2 \Rightarrow P^*H = U^*H^2K^2 \Rightarrow P^*H = U^*H^2 \Rightarrow P^* = U^*H$. Then P is nonsingular and KP^*AP is k-diagonal matrix. That is, A is congruent to a k-diagonal matrix.

Conversely, Assume that A is congruent to a k-diagonal matrix. Then there exists a nonsingular matrix P such that $KP^*AP = D$ is a k-diagonal matrix. By the polar factorization theorem of matrices, there exists a positive definite k-hermite matrix H and a k-unitary matrix U such that $P = HU \Rightarrow P^* = U^*H^* = U^*KHK \Rightarrow P^*H = U^*KHKH \Rightarrow P^*H = U^*(KH)^2 \Rightarrow P^*H = U^*H^2K^2 \Rightarrow P^*H = U^*H^2 \Rightarrow P^* = U^*H$. Thus $KU^*(HAH)U = D$ is a k-diagonal matrix. By Schur's theorem [2] and definition (3.5), A is generalized k-normal.

Theorem 3.9: Let $A \in \mathbb{C}_{n\times n}$ be a generalized k-normal matrix. Then there exists a nonsingular matrix P such that $A = KP^*\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P$, where $D = diag\,(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_r})$, r = rank(A), $-\pi \le \theta_1 \le \theta_2 \le .... \le \theta_r < \pi,\ \theta_r - \theta_1 < \pi$ and $e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_r}$ are all nonzero k-eigen values of $P^{-1}(KP^*)^{-1}A$.

Proof: Since A is generalized k-normal matrix. By Theorem (3.8) there exists a nonsingular matrix Q such that $A = KQ^*\,diag\,(\lambda_1, \lambda_2, ..., \lambda_r, 0, 0, ..., 0)Q$, where r = rank(A) is the rank of A, $\lambda_1, \lambda_2, ..., \lambda_r$ are nonzero complex numbers.

Let $\lambda_j = \rho_j e^{i\theta_j}$, $\rho_j > 0$, $j = 1, 2, ..., r$ and $-\pi \le \theta_1 \le \theta_2 \le .... \le \theta_r < \pi$.

Denote $R = diag\,(\rho_1^{1/2}, \rho_2^{1/2}, ..., \rho_r^{1/2}, 1, ..., 1)$ and $P = RQ \Rightarrow P^* = Q^*R^*$.

Then P is nonsingular and $A = KP^*\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P.$

Since $P^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP^*)^{-1}A = P^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP^*)^{-1}(KP^*)\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P$

$= P^{-1}\begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix}P,\ e^{2i\theta_1}, e^{2i\theta_2}, ..., e^{2i\theta_r}$ are all nonzero k-eigen values of $P^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP^*)^{-1}A.$

Without loss of generality, $-\pi \le 2\theta_{j1} \le 2\theta_{j2} \le .... \le 2\theta_{jr} < \pi (\text{mod}\, 2\pi)$.
Then $2\theta_r - 2\theta_1 \le 2\theta_{jr} - 2\theta_{j1} < 2\pi(\text{mod}\, 2\pi)$. Thus $\theta_r - \theta_1 < \pi$.
Since $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P$,

$$P^{-1}(KP^*)^{-1}A = P^{-1}(KP^*)^{-1}(K P^*)\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P = P^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P.$$

Hence $e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_r}$ are all nonzero k-eigen values of $P^{-1}(K P^*)^{-1}A$.

Theorem 3.10: Let $A \in \mathbb{C}_{n\times n}$ be a generalized k-normal matrix. If there exist a nonsingular matrix P and Q such that $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = K Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q$, where $D = diag(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_r}), -\pi \le \theta_1 \le \theta_2 \le .... \le \theta_r < \pi, \theta_r - \theta_1 < \pi$, r = rank(A), $\tilde{D} = diag(e^{iw_1}, e^{iw_2}, ..., e^{iw_r}), -\pi \le w_1 \le w_2 \le .... \le w_r < \pi$ and $w_r - w_1 < \pi$.
Then $\theta_j = w_j$ for $1 \le j \le r$.

Proof: Given $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = K Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q$

$$\Rightarrow \; K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = K Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q.$$

Pre multiply by K on both sides.
Therefore, $P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q$

$$\Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = (P^*)^{-1} Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} QP^{-1} \tag{2}$$

Denote $Q P^{-1} = R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$, where R_{11} is an rxr complex matrix.

$$\Rightarrow \; (Q P^{-1})^* = (P^*)^{-1} Q^* = R^* = \begin{pmatrix} R_{11}^* & R_{12}^* \\ R_{21}^* & R_{22}^* \end{pmatrix} \Rightarrow$$

$$(2) \Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_{11}^* & R_{12}^* \\ R_{21}^* & R_{22}^* \end{pmatrix}\begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} R_{11}^*\tilde{D} & 0 \\ R_{21}^*\tilde{D} & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} R_{11}^*\tilde{D}R_{11} & R_{11}^*\tilde{D}R_{12} \\ R_{21}^*\tilde{D}R_{11} & R_{21}^*\tilde{D}R_{12} \end{pmatrix}.$$

Thus $D = R_{11}^* \tilde{D} R_{11}$. Since D is nonsingular, R_{11} and R_{11}^* are also nonsingular.

Hence

$D^2 = D(D^*)^{-1} = (R_{11}^* \tilde{D} R_{11})(R_{11}^{-1} \tilde{D}^{-1} (R_{11}^*)^{-1})^* = R_{11}^* \tilde{D} (\tilde{D}^*)^{-1} (R_{11}^*)^{-1} = R_{11}^* \tilde{D}^2 (R_{11}^*)^{-1}$.

That is, D^2 and $\tilde{D}^2$ are similar. Thus D^2 and $\tilde{D}^2$ have the same k-eigen values.

That is, $e^{2i\theta_j} = e^{2iw_j}$ for $1 \le j \le r$.

Therefore, $2\theta_j \equiv 2w_j \pmod{2\pi}$ for $1 \le j \le r$ and hence $\theta_j \equiv w_j \pmod{\pi}$ for $1 \le j \le r$. By the conditions of the theorem, $\theta_j = w_j$ for $1 \le j \le r$.

Remark 3.11: Theorem (3.10) indicates that when a generalized k-normal matrix A satisfies $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P$, then $D = diag\,(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_r})$ is independent of the choice of the nonsingular matrix P, that is, the k-diagonal matrix D is only determined by A. So we have the following definitions.

Definition 3.12: Let A be a generalized k-normal matrix satisfying $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P$. Denote $D(KA) = D = diag\,(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_r})$, where r = rank(KA), $-\pi \le \theta_1 \le \theta_2 \le \dots \le \theta_r < \pi$ and $\theta_r - \theta_1 < \pi$. Then $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ is called the canonical form of the generalized k-normal matrix A in the sense of congruence and $\sigma_k(A) = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_r})$ is called the generalized k-spectrum of A.

Theorem 3.13: Let $A, B \in \mathbb{C}_{n\times n}$ be generalized k-normal matrices. Then A and B are congruent if and only if $\sigma_k(A) = \sigma_k(B)$, that is A and B have the same generalized k-spectrum.

Proof: Assume that A and B are congruent. Then there exists a nonsingular matrix Q such that $B = KQ^* AQ$.

By Theorem (3.9), there exists a nonsingular matrix P such that $A = K P^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} P$.

Thus $$B = KQ^* K P^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} PQ. \tag{3}$$

Denote $R = PQ$, then R is nonsingular. $\Rightarrow R = PKQK \Rightarrow R^* = KQ^* KP^*$.

Therefore (3) $\Rightarrow$ $B = R^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} R$ $\Rightarrow$ $KB = R^* \begin{pmatrix} D(KA) & 0 \\ 0 & 0 \end{pmatrix} R$.

By a Theorem (3.10), D(KB) = D(KA). Hence $\sigma(KA) = \sigma(KB)$ $\Rightarrow$ $\sigma_k(A) = \sigma_k(B)$. Conversely, Let $\sigma_k(A) = \sigma_k(B)$ $\Rightarrow$ $\sigma(KA) = \sigma(KB)$ $\Rightarrow$ D(KA) = D(KB) = D and there exists a nonsingular matrices P , Q such that

$$A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \tag{4}$$

and $$B = K Q^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q \tag{5}$$

Pre multiply by $(KP^*)^{-1}$ and post multiply by P^{-1} in (4). Therefore (4) $\Rightarrow$ $(KP^*)^{-1} AP^{-1} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$. Hence $B = (K Q^*)(KP^*)^{-1} AP^{-1} Q$. Therefore A and B are congruent.

Determinantal properties

Let $A \in \mathbb{C}_{n\times n}$. Alternatively, let $H(A) = \frac{1}{2}(A + KA^* K)$, the k-hermite part and $S(A) = \frac{1}{2}(A - KA^* K)$, the skew k-hermite parte. It is known that $A = H(A) + S(A)$ and $K A^* K = H(A) - S(A)$. A is called positive definite provided that H(A) is positive definite k-hermite matrix. It is easy to know that A is positive definite if and only if $\mathrm{Re}(Kx^*, Ax) > 0$ for each nonzero complex column vector $x = (x_1, x_2, ..., x_n)^T$, where Re(a) denotes the real part of a complex number ‘ a ’.

Theorem 4.1: Let $A \in \mathbb{C}_{n\times n}$ be a positive definite matrix, then A is generalized k-normal.

Proof: since A is positive definite, H(A) is nonsingular.

Then

$$KA^* K[H(A)]^{-1} A = [H(A) - S(A)][H(A)]^{-1}[H(A) + S(A)]$$
$$= \{H(A)[H(A)]^{-1} - S(A)[H(A)]^{-1}\}[H(A) + S(A)]$$
$$= \{I - S(A)[H(A)]^{-1}\}[H(A) + S(A)]$$
$$= H(A) + S(A) - S(A)[H(A)]^{-1}[H(A) + S(A)]$$
$$= H(A) + S(A) - S(A)[H(A)]^{-1} H(A) - S(A)[H(A)]^{-1} S(A)$$

$$= H(A) + S(A) - S(A) - S(A)[H(A)]^{-1}S(A)$$

$$= H(A) - S(A)[H(A)]^{-1}S(A) \tag{5}$$

$$A[H(A)]^{-1}KA^*K = [H(A) + S(A)][H(A)]^{-1}[H(A) - S(A)]$$

$$= \{I + S(A)[H(A)]^{-1}\}[H(A) - S(A)]$$

$$= H(A) - S(A) + S(A)[H(A)]^{-1}[H(A) - S(A)]$$

$$= H(A) - S(A) + S(A) - S(A)[H(A)]^{-1}S(A)$$

$$= H(A) - S(A)[H(A)]^{-1}S(A) \tag{6}$$

From (5) and (6), we get

$$KA^*K[H(A)]^{-1}A = A[H(A)]^{-1}KA^*K \tag{7}$$

Since H is a positive definite k-hermite matrix. Hence $[H(A)]^{-1}$ is a positive definite k-hermite matrix. By a Theorem (3.7), $X = K[H(A)]^{-1} = [H(A)]^{-1}K$ is a positive definite k-hermite matrix. Therefore (7)$\Rightarrow$ $KA^*XA = AXA^*K$. Hence A is generalized k-normal matrix.

Theorem 4.2: Let 'n' be an integer with $n \geq 3$ and $A \in \mathbb{C}_{n\times n}$ a generalized k-normal matrix. Then $\left|\det(A)\right| \geq \left|\det H(A)\right| + \left|\det S(A)\right|$. Further, the equality holds if and only if r = rank(A)<n or A is a k-hermite matrix or A is a skew k-hermite matrix.

Proof: There exists a nonsingular matrix P such that $A = K\,P^*\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P$. It is easy to see that $H(A) = KP^*H(D)P$ and $S(A) = KP^*S(D)P$. Thus rank [H(A)]=rank [H(D)] and rank [S(A)]= rank[S(D)]. We distinguish the following cases.

Case (1): If rank (A)<n. Than rank [H(A)]=rank [H(D)] $\leq$ rank (A)<n and rank [S(A)]= rank[S(D)] $\leq$ rank (A)<n. Thus $\det(A) = \det H(A) = \det S(A) = 0$ and so $\left|\det(A)\right| = \left|\det H(A)\right| + \left|\det S(A)\right|$.

Case (2): If rank (A)=n. Now, $D = diag\,(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n})$,
$\Rightarrow$ $D = diag(\cos\theta_1 + i\sin\theta_1, \cos\theta_2 + i\sin\theta_2, ..., \cos\theta_n + i\sin\theta_n)$.
$\Rightarrow$ $H(D) = diag(\cos\theta_1, \cos\theta_2, ..., \cos\theta_n)$, $S(D) = i\,diag(\sin\theta_1, \sin\theta_2, ..., \sin\theta_n)$ and $A = KP^*DP$. Then $\det(A) = (\det KP)^2 \prod_{j=1}^{n} e^{i\theta_j}$. $\Rightarrow$ $\det H(A) = (\det KP)^2 \prod_{j=1}^{n}\cos\theta_j$ and $\det S(A) = i^n(\det KP)^2 \prod_{j=1}^{n}\sin\theta_j$.

Sub case (2.1): If $\cos\theta_j = 0$ for some j.

Then $\left|\det H(A)\right| + \left|\det S(A)\right| = \left|\det KP\right|^2 \prod_{j=1}^{n}\left|\sin\theta_j\right| \le \left|\det KP\right|^2 \le \left|\det(A)\right|$. Further, the above equality holds if and only if $\left|\sin\theta_j\right| = 1$ for all j, if and only if $\cos\theta_j = 0$ for all j, if and only if H(A)=0 if and only if A is skew k-hermitian.

Sub case (2.2): If $\sin\theta_j = 0$ for some j.

Then $\left|\det H(A)\right| + \left|\det S(A)\right| = \left|\det KP\right|^2 \prod_{j=1}^{n}\left|\cos\theta_j\right| \le \left|\det KP\right|^2 \le \left|\det(A)\right|$. Further, the above equality holds if and only if $\left|\cos\theta_j\right| = 1$ for all j, if and only if $\sin\theta_j = 0$ for all j, if and only if S(A)=0 if and only if A is k-hermitian.

Sub case (2.3): If $\cos\theta_j \ne 0$ and $\sin\theta_j \ne 0$ for all j.

Now $\left|\det H(A)\right| + \left|\det S(A)\right| = \left|\det KP\right|^2 \left(\prod_{j=1}^{n}\left|\cos\theta_j\right| + \prod_{j=1}^{n}\left|\sin\theta_j\right| \right)$

$\Rightarrow \left|\det H(A)\right| + \left|\det S(A)\right| = \left|\det KP\right|^2 \left(\left|\cos\theta_1 \cos\theta_2\right| + \left|\sin\theta_1 \sin\theta_2\right| \right) \le \left|\det KP\right|^2 = \left|\det(A)\right|$

By summing up the above cases the proof of this theorem is completed.

Remark 4.3: Let $x_1, x_2, ..., x_n, y_1, y_2, ...y_n$ be non negative real numbers. Then $(x_1, x_2, ..., x_n)^{\frac{1}{n}} + (y_1, y_2, ...y_n)^{\frac{1}{n}} \le (x_1 + y_1)^{\frac{1}{n}} (x_2 + y_2)^{\frac{1}{n}} ... (x_n + y_n)^{\frac{1}{n}}$ and the equality holds if and only if there exists and integer j such that $x_j = y_j = 0$ or $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ...y_n)$ are linearly dependent.

Theorem 4.4: Let 'n' be an integer with $n \ge 3$ and $A \in \mathbb{C}_{n\times n}$ a generalized k-normal matrix with generalized k-spectrum $\sigma_k(A) = (e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_r})$, where r = rank(A). Then $\left|\det(A)\right|^{\frac{2}{n}} \ge \left|\det H(A)\right|^{\frac{2}{n}} + \left|\det S(A)\right|^{\frac{2}{n}}$. Further, the equality holds if and only if r<n or $(\cos^2\theta_1, \cos^2\theta_2, ..., \cos^2\theta_n)$ and $(\sin^2\theta_1, \sin^2\theta_2, ..., \sin^2\theta_n)$ are linearly dependent.

Proof: There exists a nonsingular matrix P such that $A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P.$ It is easy to see that $H(A) = KP^* H(D) P$ and $S(A) = KP^* S(D) P$. Thus rank [H(A)]=rank [H(D)] and rank [S(A)]= rank[S(D)]. We distinguish the following cases.

Case (1): If rank (A)<n.

Than rank [H(A)]=rank [H(D)] $\le$ rank (A)<n and rank [S(A)]= rank[S(D)] $\le$ rank

(A)<n.

Thus $\det(A) = \det H(A) = \det S(A) = 0$.

There fore $\left|\det(A)\right|^{\frac{2}{n}} \geq \left|\det H(A)\right|^{\frac{2}{n}} + \left|\det S(A)\right|^{\frac{2}{n}}$.

Case (2): If rank (A)=n. Now, $D = diag\,(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n})$,

$\Rightarrow \quad H(D) = diag(\cos\theta_1, \cos\theta_2, ..., \cos\theta_n)$, $S(D) = i\,diag(\sin\theta_1, \sin\theta_2, ..., \sin\theta_n)$ and $A = KP^* DP$. Then $\det(A) = (\det KP)^2 \prod_{j=1}^{n} e^{i\theta_j}$. $\Rightarrow$ $\det H(A) = (\det KP)^2 \prod_{j=1}^{n} \cos\theta_j$ and $\det S(A) = i^n (\det KP)^2 \prod_{j=1}^{n} \sin\theta_j$.

Thus $\left|\det H(A)\right| + \left|\det S(A)\right| = \left|\det KP\right|^2 \left(\prod_{j=1}^{n} \cos\theta_j + \prod_{j=1}^{n} \sin\theta_j \right)$

$$\Rightarrow \quad \left|\det H(A)\right|^{\frac{2}{n}} + \left|\det S(A)\right|^{\frac{2}{n}} = \left|\det KP\right|^{\frac{4}{n}} \left(\prod_{j=1}^{n} (\cos^2\theta_j)^{\frac{1}{n}} + \prod_{j=1}^{n} (\sin^2\theta_j)^{\frac{1}{n}} \right).$$

By Remark (4.3), $\left|\det H(A)\right|^{\frac{2}{n}} + \left|\det S(A)\right|^{\frac{2}{n}} \leq \left|\det(A)\right|^{\frac{2}{n}}$ and the equality holds if and only if there exists an integer j such that $\cos^2\theta_j = \sin^2\theta_j = 0$ or $(\cos^2\theta_1, \cos^2\theta_2, ..., \cos^2\theta_n)$ and $(\sin^2\theta_1, \sin^2\theta_2, ..., \sin^2\theta_n)$ are linearly dependent. Since $\cos^2\theta_j + \sin^2\theta_j = 1$, the equality holds if and only if $(\cos^2\theta_1, \cos^2\theta_2, ..., \cos^2\theta_n)$ and $(\sin^2\theta_1, \sin^2\theta_2, ..., \sin^2\theta_n)$ are linearly dependent. Summing up the above cases, we finish the proof of this theorem.

References

[1] Hong-ping, M. A., Zheng-ke, MIAO, and Jiong-sheng, L. I., 2008, “Generalized normal matrix”, Appl. Math. J.Chinese univ., 23(2), pp.240-244.

[2] Horn, R. A., and Johnson, C. R., 1985, Matrix Analysis, Cambridge: Cambridge University press, pp.79-482.

[3] Beckenbach, E. F., and Bellman, R., 1983 “ Inequalities,” Berlin: Springer, PP.26-

[4] Krishnamoorthy, S., and Subash, R., 2011 “On k-normal matrices” International J. of Math. Sci. & Engg. Appls. Vol.5, no.II , pp.119-130.