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Continuous Indexed Variable Systems

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Abstract

In this paper, we introduce the theory of Continuous Indexed Variable Systems as a new structure of Indexed Variable Systems. Also, we study it with the view of Topology and Algebra and we prove some essential properties about the category of Indexable sets.

Keywords: Continuous Indexed Variable System, Category, Covariant, Contravariant, Functor, Indexable set, Local Topology.

Introduction

An idea for introducing Indexed Variable Systems (IVS for short) can be found in [3], [6]. According this, an IVS contains a nonempty set X and a family of membership-congruent relations $\{\Xi_r\}_{r\in R}$, where R is a subset of interval [0,1] and for each $x, y \in X$ the following conditions hold:

(i) there exists $r \in R$ such that $x =_r y$,

(ii) if $x =_r y$, then $y =_r x$,

(iii) $x =_1 y$ if and only if x and y are not two different objects ; viz, $\{x, y\}$ is unitary.

Moslemy and Pouhassani [3] proved that each IVS is a metric space and vice versa, furthermore, they proved that Indexed Identity relation is not an equivalence relation. Also they introduced the concepts of local topology and topological Entropy for an IVS and provided some properties about them in [4]. According this properties, Pouhassani [5] gave a new structure for ISG that can be useful for further investigations in the future.

In the following sections, we will define Continuous Indexed Variable Systems (CIVS for short) with respect to previous studies and meanwhile refined previous results, some properties and consequences in this topic –specially algebraic properties - will be presented.

Indexable Sets

We start this section by the following definition that expresses the definition of IVS with functions.

Definition II.1. Let X be an arbitrary nonempty set. A function $f : X \times X \to [0,1]$ is called Continuous Indexed Variable System (CIVS for short) over X if the following conditions hold for each $x, y \in X$.

i. There exists $r \in [0,1]$ such that $f(x,y) = r$.
ii. $f(x,y) = f(y,x)$.
iii. $f(x,y) = 1$ if and only if $x = y$.

If there exists such function , then X is called an Indexable set. Here we present some examples of such systems.

Example II.2. (1) Let $X = (0,\infty)$, then the function

$$f(x,y) = \frac{2xy}{x^2 + y^2} \text{ is a CIVS over } X .$$

(2) Let $X = (-\frac{\pi}{4}, \frac{\pi}{4})$, then the function $f(x,y) = Cos\,|x - y|$ is a CIVS over X .

(3) $f(x,y) = \dfrac{1}{|x - y| + 1}$ is a CIVS over $\mathbb{R}$.

In the rest of this section, we assume that X is an arbitrary nonempty set.

Lemma II.3. If f and g are CIVSs over X then so is $f.g$, where $(f.g)(x,y) = f(x,y).g(x,y)$ for each $x, y \in X$. Also every finite product of CIVSs over X is CIVS over it.

Proof. The first assertion easily follows from Definition II.1 and the second one, can be deduce by several times using the first part. ■

Lemma II.4. Let f be a CIVS over X and x, y arbitrary and fixed elements of X . Consider the sequence $(f^n(x,y))_{n=1}^{\infty}$, where $f^n(x,y) = f(x,y).f^{n-1}(x,y)$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} f^n(x,y) = \delta(x,y)$, where $\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

Proof. It is trivial by the Definition II.1. ■

Lemma II.5. Let $f:[0,1]\to[0,1]$ be an injective function and g be a CIVS over X. Then fog is a CIVS over X if and only if $f(1)=1$.

Proof. If fog is a CIVS over X, then $x=y$ implies that $fog(x,y)=1$ and so $g(x,y)=f^{-1}(1)$. In the other hand, $x=y$ implies that $g(x,y)=1$, thus $f^{-1}(1)=1$. Vice versa, the first two conditions of CIVS's definition hold immediately for fog over X. Also for each $x,y\in X$, we have $fog(x,y)=1$ if and only if $g(x,y)=1$. It follows that fog satisfies in the third condition of CIVS's definition. ■

Example II.6. In the second part of Example II.2, one has $g(x,y)=\cos|x-y|$ is a CIVS over $X=(\frac{-\pi}{4},\frac{\pi}{4})$. Let $f:[0,1]\to[0,1]$ such that $f(x):=\frac{x+1}{2}$. Hence $\frac{\cos|x-y|+1}{2}$ is a CIVS over X by Lemma II.5.

Local Topologies on the Indexable Sets

In this section, X be an arbitrary nonempty set and f is a CIVS over it. Then X is an Indexable set. For each $x\in X$ and $t\in[0,1]$, we denote

$$N_t^f(x)=\{y\in X; f(x,y)\succ t\}.$$

By using this notation, we construct a topology for X in the next result.

Theorem III.1. Let $x\in X$ and $\tau_f(x)$ consists of all sets $N_t^f(x)$, where $t\in[0,1]$. Then X is a topological space with $\tau_f(x)$.

Proof. It is trivial that $\tau_f(x)$ consists X and empty set. Suppose that $\{N_{t_i}^f(x)\}_{i\in I}$ be an arbitrary subset of $\tau_f(x)$. Set $t:=\inf_{i\in I} t_i$. One can see that

$$N_t^f(x)=\bigcup_{i\in I} N_{t_i}^f(x)\in\tau_f(x).$$

Also, if $\{N_{t_i}^f(x)\}_{i=1}^n$ be a finite subset of $\tau_f(x)$, then

$$\bigcap_{i=1}^n N_{t_i}^f(x)=N_s^f(x)\in\tau_f(x),$$

where $s=\sup_{1\le i\le n} t_i$. Therefore $\tau_f(x)$ is a topology on X. ■

The above topology proposed as local topology generated by x (or depends on (x)).

Theorem III.2. Let Y be an arbitrary nonempty set and $\phi : X \to Y$ is an bijective function. Then Y is an Indexable set and ϕ is a homomorphism between local topologies generated by x and $\phi(x)$ for each $x \in X$ (i.e. ϕ and its inverse are both continuous).

Proof. Define the function $g : Y \times Y \to [0,1]$ by $g(y_1,y_2) = f(x_1,x_1)$, where $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then g is a CIVS over Y, because

$$g(y_1,y_2) = f(x_1,x_2)$$
$$= f(x_2,x_1)$$
$$= g(y_2,y_1)$$

and

$$g(y_1,y_2) = 1 \Leftrightarrow f(x_1,x_2) = 1$$
$$\Leftrightarrow x_1 = x_2$$
$$\Leftrightarrow y_1 = y_2 .$$

For the second assertion, let $x_0 \in X$, $\phi(x_0) = y_0$ and $t \in [0,1]$. Then

$$\phi(N_t^f(x_0)) = \phi(\{x \in X ; f(x_0,x) \succ t\})$$
$$= \{\phi(x) ; f(x_0,x) \succ t\}$$
$$= \{y \in Y ; g(y_0,y) \succ t\}$$
$$= N_t^g(y_0)$$

implies that ϕ is continuous. Similarly, ϕ^{-1} is also continuous and it completes the proof. ■

Remark III.3. In the Theorem III.2, if $X = Y$, then local topologies $\tau_f(x)$ and $\tau_g(\phi(x))$ are homomorphic.

Theorem III.4. Let X be a finite dimension vector space on the field of real numbers $\mathbb{R}$. Assume that $A = \{a_1,...,a_n\}$ be a nonempty subset of X and L_A denotes the subspace of X that is generated by A. Then L_A is a Indexable set.

Proof. Since every generator set of a vector space contains a basis, then without loss of generality, we may and do assume that A is a base for L_A. We define $\theta: L_A \times L_A \to [0,1]$, such that

$$\theta(\sum_{i=1}^{n} r_i a_i, \sum_{i=1}^{n} r_i' a_i) := \inf\{f\ (r_i a_i, r_i' a_i); i = 1,...,n\}$$

for each $r_i, r_i' \in \mathbb{R}$.

We show that it is a CIVS over L_A. It's clear that θ is well defined and satisfies in the first and second conditions from the definition of CIVS. Then, for completing the proof, we need to show that for each element u,v of L_A, $\theta(u,v) = 1$ if and only if $u = v$. As $u,v \in L_A$, there exists $r_i, r_i' \in \mathbb{R}$ such that $u = \sum_{i=1}^{n} r_i a_i$ and $v = \sum_{i=1}^{n} r_i' a_i$. Since A is a basis for L_A and f is a CIVS over X, we have

$$\theta(u,v) = 1 \Leftrightarrow f\ (r_i a_i, r_i' a_i) = 1 \ \ \forall i = 1,...,n$$

$$\Leftrightarrow r_i a_i = r_i' a_i \ \ \forall i = 1,...,n$$

$$\Leftrightarrow r_i = r_i' \ \ \forall i = 1,...,n$$

$$\Leftrightarrow u = v .$$

Then L_A is a Indexable set, as claimed. ■

The Category of Indexable Sets

In this section, we study Indexable sets from an Algebraic point of view. So, we construct the category of Indexable sets which is a useful language and provides a field for more studies about Indexable sets and CIVSs. For more information about categories, we refer the reader to [1], [2].

Definition IV.1. A category is a class C of objects such as $A, B, C, ...$ and a disjoint family of morphisms for each element A, B of C which is denoted by $Hom_C(A,B)$ (elements of $Hom_C(A,B)$ are considered as $f: A \to B$) such that there is a function for each element A, B, D of C such as

$$\theta_{ABD} : Hom_C(A,B) \times Hom_C(B,D) \to Hom_C(A,D)$$

that $\theta_{ABD}(f, g) = gof$, where the composition gof is defined and it satisfies in the following conditions:

i. for each $f: B \to D, g: A \to B$ and $h: E \to A$, we have $(fog)oh = fo(goh)$ (Associativity);

ii. for each element of C such as B , there exists $1_B \in Hom_C(B,B)$ such that for $f:A \to B$ and $g:B \to A$, one has $1_B o f = f$ and $g o 1_B = g$ (Identity).

Definition Iv.2. Let B and C be two categories. B is called a subcategory of C if it satisfies in the following condition:

i. each object of B is a object of C ;

ii. for each object X and Y in B , $Hom_B(X,Y) \subseteq Hom_C(X,Y)$;

iii. the combination of morphisms in B and C are exactly the same;

iv. for each object of B , the identity morphisms in B and C are exactly the same.

Also, the subcategory B of C is called full subcategory if for all X and Y in B , one has

$$Hom_B(X,Y) = Hom_C(X,Y) .$$

Example and Notation IV.3. Let S be a class of all sets. For each $A,B \in S$, assume that $Hom_S(A,B)$ is the set of all functions $f:A \to B$ and the combination between functions is the same usual combination. Then S is a category. One can see that the class of all Indexable sets is a full subcategory of S that we denote it by S^*.

The following Lemma shows that the set of all morphisms between two Indexable sets is Indexable set, too.

Lemma IV.4. Let $X \in S$ and $Y \in S^*$. Then $Hom_{S^*}(X,Y) \in S^*$.

Proof. Let f be a CIVS over Y . We define

$$h: Hom_{S^*}(X,Y) \times Hom_{S^*}(X,Y) \to [0,1]$$

such that $h(\alpha,\beta) := \inf\{f(\alpha(x),\beta(x)); x \in X\}$ for all $\alpha,\beta \in Hom_{S^*}(X,Y)$. Then by the similar proof of Theorem III.3, the assertion is followed. ■

One of the interesting properties of categories is existence of product in them. We show that the product exists for each family of objects in S^* . For this mean, we need the following definition.

Definition IV.5. Let C be a category and $\{A_i; i \in I\}$ a family of objects in C . A product for $\{A_i; i \in I\}$ is an object P of C with a family of morphisms such as $\{\varphi_i : P \to A_i; i \in I\}$ such that for each object B of C and family $\{\psi_i : B \to A_i; i \in I\}$ of morphisms , there exists a unique morphism $\phi : B \to P$ such that $\varphi_i o \phi = \psi_i$ for each $i \in I$.

Theorem IV.6. Each family of objects in S^* has a product.

Proof. Assume that $\{A_i; i \in I\}$ is a family of objects in S^* and $\prod_{i \in I} A_i$ is a Cartesian product of them (we recall that $\prod_{i \in I} A_i = \{(a_i)_{i \in I}; a_i \in A_i, i \in I\}$).

Then it is a product in category S by [1]. Since S^* is a full subcategory of S, it's sufficient to prove that $\prod_{i \in I} A_i$ is an Indexable set. For this mean, we define

$$h : \prod_{i \in I} A_i \times \prod_{i \in I} A_i \to [0,1]$$

Such that $h((a_i)_{i \in I}, (b_i)_{i \in I}) := \inf\{h_i(a_i, b_i); i \in I\}$, where h_i is a CIVS over A_i and $a_i, b_i \in A_i$ for each $i \in I$. As $h_i(a_i, b_i) = h_i(b_i, a_i)$ for all $a_i, b_i \in A_i$, then $h((a_i)_{i \in I}, (b_i)_{i \in I}) = h((b_i)_{i \in I}, (a_i)_{i \in I})$.

Also we have $h((a_i)_{i \in I}, (b_i)_{i \in I}) = 1$ if and only if $h_i(a_i, b_i) = 1$ if and only if $a_i = b_i$ for each $i \in I$. Thus h is a CIVS on $\prod_{i \in I} A_i$. ■

Remark IV.7. If product exists for a family of objects in an arbitrary category, then it is unique up to isomorphism, see [1].

Finally, we conclude this paper by defining maps that preserve the structure of Indexable sets. For this mean, we need the definition of covariant and contravarient functors.

Definition IV.8. Assume that C and D are two arbitrary categories. A covariant (resp. contravariant) functor is a pair of functions that both of them are denoted by T such that one of them gives to each object A of C, one object $T(A)$ of D and the other gives to each morphism $f : A \to B$ of C a morphism such as $T(f) : T(A) \to T(B)$ (resp. $T(f) : T(B) \to T(A)$) of D that:

i. for each object A of C, $T(1_A) = 1_{T(A)}$;
ii. for each two morphisms f and g of C that their combination can be defined, one has $T(gof) = T(g)oT(f)$ (resp. $T(gof) = T(f)oT(g)$).

Theorem IV.9. (a) Let $X \in S$. Then $h_X(.) : S^* \to S^*$ is defined by

$$h_X(Y) := Hom_{S^*}(X, Y),$$

and

$$h_X(f)(g) := fog$$

for all $Y, Z \in S^*$, $Hom_{S^*}(Y, Z)$ and $g \in Hom_{S^*}(X, Y)$ is a covariant functor.

(b) Let $Y \in S^*$. Then $h^Y(.) : S^* \to S^*$ is defined by

$$h^Y(X) := Hom_{S^*}(X, Y)$$

and

$$h^Y(f)(g) := gof$$

for each $X, Z \in S^*$, $Hom_{S^*}(X, Z)$ and $g \in Hom_{S^*}(Z, Y)$ is a contravarient functor.

Proof. (a) By Lemma IV.4, $h_X(.)$ is well defined. Also, by the definition of $h_X(.)$, one can see that $h_X(1_Y) = 1_{h_X(Y)}$ for each $Y \in S^*$. Hence, we just show that

$$h_X(gof) = h_X(g)oh_X(f),$$

Where $g : Y \to T, f : Z \to Y$ and $Y, Z, T \in S^*$.
We have

$$h_X(gof) : Hom_{S^*}(X, Z) \to Hom_{S^*}(X, T)$$

that

$$h_X(gof)(h) = (gof)oh$$
$$= go(h_X(f)(h))$$
$$= (h_X(g)oh_X(f))(h)$$

for each $h \in Hom_{S^*}(X, Z)$. Thus

$$h_X(gof) = h_X(g)oh_X(f)$$

and this completes the proof.

(b) By using contravariant functor instead of covariant functor , the proof proceed exactly in (a), so we leave it to the reader. ■

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