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# Distribution of the Number of Times M/M/2/N Queuing System Reaches its Capacity in Time t under Catastrophic Effects

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**Abstract**

This paper presents the distribution of the number of times that a limited capacity two servers Markovian queue subjected to catastrophes reaches its capacity in time t. The occurrence of a catastrophe makes the system empty instantaneously. After the occurrence of a catastrophe both the servers become ready to serve the new customers. The afore said distribution is obtained as a marginal distribution of the joint distribution of the number of customers in the system at time t and the number of times system reaches its capacity in time t. Some measures of effectiveness are obtained explicitly with catastrophic effects.

**Keywords:** Catastrophes, homogeneous servers, markovian queue, p.g.f.

## Introduction

In the last one decade a lot of work has been done by various authors taking into consideration the concept of catastrophe Kumar *et. al.* [5] studied the transient behaviour of the M/M/2 queue with catastrophes. Crescenzo *et. al.* [2] made a continuous approximation of M/M/1 queue with catastrophe. Chao *et. al* [9] obtained the transient analysis of immigration birth-death processes with total catastrophe. In all the mentioned studies the researchers have obtained the state probabilities in one way or the other and have computed various measures of performance. In this paper we have explicitly obtained the distribution of the number of times the system prone to catastrophes reaches its capacity in time t. various other measures of performance are obtained explicitly.

## Model Description with Catastrophe

We consider an M/M/2/N queueing system having two homogenous servers with FCFS discipline subjected to catastrophe. The customers arrive at a counter in accordance with a Poisson process with mean arrival rate $\lambda>0$. Each server serves one customer at a time if available. The service time distribution of a customer is negative exponential with mean rate $\mu>0$. The queuing process starts at time zero with zero state of the system. Catastrophes occur according to Poisson process with mean rate $\xi$ only when the system is not empty. The occurrence of catastrophe makes the queuing system empty instantaneously, and simultaneously both the servers become ready to accept the new customers.

We define

$$P_{m,n}(t) = \text{Prob. } [X(t) = m, Y(t) = n],\ 0 \le n \le N \tag{1}$$

Where

X (t) = the number of times the system reaches its capacity in time t.

Y (t) = the number of customers in the system at time t.

The marginal probabilities are

$$P_{.,n}(t) = \sum_{m=0}^{\infty} P_{m,n}(t) \quad \text{and} \quad P_{m,.}(t) = \sum_{n=0}^{N} P_{m,n}(t)$$

### Differential-Difference Equations Governing the System

$$\frac{d}{dt}P_{m,0}(t) = -(\lambda+\xi)P_{m,0}(t) + \mu P_{m,1}(t) + \xi P_{m,.}(t) \quad ; n = 0, m \ge 0 \tag{2}$$

$$\frac{d}{dt}P_{m,n}(t) = -(\lambda+2\mu+\xi)P_{m,n}(t) + \lambda P_{m,n-1}(t) + 2\mu P_{m,n+1}(t) \quad ; 1 < n < N, m \ge 0 \tag{3}$$

$$\frac{d}{dt}P_{m,1}(t) = -(\lambda+\mu+\xi)P_{m,1}(t) + \lambda P_{m,0}(t) + 2\mu P_{m,2}(t) \quad ; n = 1, m \ge 0 \tag{4}$$

$$\frac{d}{dt}P_{m,N}(t) = -(2\mu+\xi)P_{m,N}(t) + \lambda P_{m-1,N-1}(t) \quad ; n = N, m \ge 1 \tag{5}$$

Taking Laplace transform of the equations [2]-[5] w.r.t t we have

$$(s+\lambda+\xi)\bar{P}_{0,0}(s) = 1+\mu\bar{P}_{0,1}(s)+\xi\bar{P}_{0,.}(s); n=0, m=0 \qquad [6]$$

$$(s+\lambda+\xi)\bar{P}_{m,0}(s) = \mu\bar{P}_{m,1}(s)+\xi\bar{P}_{m,.}(s); n=0, m>0 \qquad [7]$$

$$(s+\lambda+\mu+\xi)\bar{P}_{m,1}(s) = \lambda\bar{P}_{m,0}(s)+2\mu\bar{P}_{m,2}(s); n=1 \qquad [8]$$

$$(s+\lambda+2\mu+\xi)\bar{P}_{m,n}(s) = \lambda\bar{P}_{m,n-1}(s)+2\mu\bar{P}_{m,n+1}(s); 2\le n<N \qquad [9]$$

$$(s+2\mu+\xi)\bar{P}_{m,N}(s) = \lambda\bar{P}_{m-1,N-1}(s); n=N \qquad [10]$$

Since

$P_{0,0}(0) = 1$

Where

$$\bar{P}_{m,n}(s) = \int_0^\infty e^{-st}P_{m,n}(t)dt$$

Define the probability generating functions by

$$\bar{P}_n(x,s) = \sum_{m=0}^{\infty}\bar{P}_{m,n}(s)x^m \qquad [11]$$

$$\bar{H}(x,y;s) = \sum_{n=0}^{N}\bar{P}_n(x,s)y^n \qquad [12]$$

$$\bar{P}_.(x,s) = \sum_{m=0}^{\infty}\bar{P}_{m,.}(s)x^m \qquad [13]$$

Multiplying equation [6] to [10] by $x^m$, summing over the ranges of m and using [11], we have

$$(s+\lambda+\xi)\bar{P}_0(x,s) = 1+\mu\bar{P}_1(x,s)+\xi\bar{P}_.(x,s); n=0 \qquad [14]$$

$$(s+\lambda+\mu+\xi)\bar{P}_1(x,s) = \lambda\bar{P}_0(x,s)+2\mu\bar{P}_2(x,s); n=1 \qquad [15]$$

$$(s+\lambda+2\mu+\xi)\bar{P}_n(x,s) = \lambda\bar{P}_{n-1}(x,s)+2\mu\bar{P}_{n+1}(x,s); n=2,3,...,N-1 \qquad [16]$$

$$(s+2\mu+\xi)\bar{P}_N(x,s) = \lambda x\bar{P}_{N-1}(x,s); n=N \qquad [17]$$

Multiplying equation [14] to [17] by $y^n$, summing over the ranges of n and using [12, 13], we have on simplification:

$$\bar{H}(x,y;s)=\frac{y[s+\xi(2-y)]-s(1-y)[2\mu+y(s+\lambda+\xi)]\bar{P}_0(x,s)+\lambda\begin{bmatrix} sy(x-1)(2-y)+ \\ sy^{N+1}\left\{(x-1)+\dfrac{\lambda x(1-y)}{s+2\mu+\xi}\right\}\end{bmatrix}\bar{P}_{N-1}(x,s)}{s[(s+\xi)y+(1-y)(\lambda y-2\mu)]} \qquad [18]$$

Since

$$\bar{P}_N(x,s)=\frac{\lambda x}{s+2\mu+\xi}\bar{P}_{N-1}(x,s)$$

The zeros of the denominator in [18] are given by

$$\alpha_i(s)=\frac{(s+\lambda+2\mu+\xi)\pm\sqrt{(s+\lambda+2\mu+\xi)^2-4\times 2\lambda\mu}}{2\lambda},i=1,2. \qquad [19]$$

The existence of $\bar{H}(x,y;s)$ is only possible if numerator vanishes for α1 and α2 the two zeros of the denominator. This will give rise two equations, solving them we have:

$$\bar{P}_0(x,s)=\frac{\bar{A}_1(x,s)}{\bar{A}(x,s)} \qquad [20]$$

$$\bar{P}_{N-1}(x,s)=\frac{\bar{A}_2(.,s)}{\bar{A}(x,s)} \qquad [21]$$

Where

$$\bar{A}_1(x,s)=2\mu\lambda\left[\frac{\lambda x}{s+2\mu+\xi}\left[\{V(N+1)-V(N)\}-\xi\gamma^{-2}\{V(N)-V(N-1)\}\right]+(1-x)\left[s\{V(1)-V(N)\}-\xi\{V(N)-\gamma^{-2}V(N-1)\}\right]\right]$$

$$\bar{A}_2(.,s)=2\mu\left[(s+\xi)(s+2\lambda)+2\mu\xi\right]V(1)$$

$$\bar{A}(x,s)=2\mu\lambda^2 s\begin{bmatrix}(1-x)\left[2V(1)+\{V(N+1)-\gamma^{-2}V(N)\}+\dfrac{s+\lambda+\xi}{\lambda}\{V(N)-\gamma^{-2}V(N-1)\}\right] \\ +\dfrac{x}{s+2\mu+\xi}\left[\lambda\{V(N+2)-\gamma^{-2}V(N+1)\}+(s+\xi)\{V(N+1)-\gamma^{-2}V(N)\}-(s+\lambda+\xi)\{V(N)-\gamma^{-2}V(N-1)\}\right]\end{bmatrix}$$

$$V(r)=\alpha_1^r(s)-\alpha_2^r(s);r=0,1,2,.......$$

$$\alpha_1\alpha_2=\frac{2\mu}{\lambda}(=\gamma^{-2}\ say)$$

If we write

$$(s+\xi)y+(1-y)(\lambda y-2\mu)=-\lambda(\alpha_1-y)(\alpha_2-y)$$

$$(\alpha_1-y)^{-1}=\alpha_1^{-1}\sum_{n=0}^{\infty}(y/\alpha_1)^n$$

$$(\alpha_2-y)^{-1}=\alpha_2^{-1}\sum_{n=0}^{\infty}(y/\alpha_2)^n$$

[18] Yields

$$\frac{y[s+\xi(2-y)]-s(1-y)[2\mu+y(s+\lambda+\xi)]\bar{P}_0(x,s)+\lambda\begin{bmatrix} sy(x-1)(2-y)+ \\ sy^{N+1}\left\{(x-1)+\dfrac{\lambda x(1-y)}{s+2\mu+\xi}\right\}\end{bmatrix}\bar{P}_{N-1}(x,s)}{s\lambda V(1)}$$

$$\times\left[\alpha_1^{-1}\sum_{n=0}^{\infty}\left(\frac{y}{\alpha_1}\right)^n-\alpha_2^{-1}\sum_{n=0}^{\infty}\left(\frac{y}{\alpha_2}\right)^n\right] \qquad [22]$$

Now $\bar{P}_n(x,s)$ is the coefficient of $y^n$ in [12]. Comparing the coefficients of $y^n$ on both sides of [22], we have:

$$\bar{P}_n(x,s)=\frac{\gamma^{2n}}{s\lambda V(1)}\left[\{2\mu s\gamma^2T(n)+s(s+\lambda+\xi)T(n-1)\}\bar{P}_0(x,s)-\xi T(n-1)-(s+\xi)V(n)\right],0\le n\le N \qquad [23]$$

Where

T (n) = V (n+1) - $\gamma^{-2}$ V (n)

Since $\bar{P}_0(x,s)$ is given by [20], so $\bar{P}_n(x,s)$ are explicitly known. For setting x=0, [23] gives

$$\bar{P}_{0,n}(s)=\frac{\gamma^{2n}}{s\lambda V(1)}\left[\frac{\left\{2\mu\gamma^2T(n)+(s+\lambda+\xi)T(n-1)\right\}\left\{s[V(1)-V(N)]-\xi T(N-1)\right\}}{\left\{2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)\right\}}-(s+\xi)V(n)-\xi T(n-1)\right] \qquad [24]$$

Applying the Leibniz differentiation theorem to [20], setting x=0 and dividing both sides by m!, we have:

$$\bar{P}_{m,0}(s)=\left[\frac{\left\{sV(1)-(s+\xi)V(N)+\xi\gamma^{-2}V(N-1)\right\}\left\{\begin{matrix}2\lambda\gamma^2(s+2\mu+\xi)V(1)-\gamma^2\lambda^2T(N+1)-(s-\lambda+2\mu+\xi)\\ [(s+\xi)\gamma^2T(N-1)-\lambda V(N-1)]\end{matrix}\right\}\times\left\{2\lambda\gamma^2[(s+2\mu+\xi)V(1)+\mu T(N)]+\gamma^2T(N-1)[(s+\lambda+\xi)^2+2\mu(s+\lambda+\xi)]-\gamma^2\lambda^2V(N+2)\right\}^{m-1}}{s\gamma^{2m}(s+2\mu+\xi)^m\left\{2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)\right\}^{m+1}}\right] \quad [25]$$

Whence form [23]

$$\bar{P}_{m,n}(s)=\frac{\gamma^{2n}}{s\lambda V(1)}\left[\left\{2\mu s\gamma^2T(n)+s(s+\lambda+\xi)T(n-1)\right\}\bar{P}_{m,0}(s)-(s+\xi)V(n)-\xi T(n-1)\right],m\geq 1 \quad [26]$$

We know that

$$(1-a/b)^{-(j+1)}=\sum_{i=0}^{\infty}\left(C_i^{j+i}\right)\frac{a^i}{b^i},(a+b)^{j+1}=a(a+b)^j+b(a+b)^j \textit{ and } (a-b)^j=\sum_{i=0}^{j}(-1)^i\left(C_i^j\right)a^{j-i}b^i$$

Using these Identities in [24] and [26], we have

$$\bar{P}_{0,n}(s)=\gamma^n\nu^{-1}\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}2^{-(j+1)}\lambda^{-(j-l+2)}\gamma^{-2(j+k-l)}\left\{2\mu\gamma^{-1}\right.\right.$$

$$\left([R^{-(g+j+h-l-n-1)}-R^{-(g+j+h-l+n+1)}]-\gamma^{-1}[R^{-(g+j+h-l-n)}-R^{-(g+j+h-l+n)}]\right)$$

$$+(s+\lambda+\xi)^{-(k-l)}\left([R^{-(g+j+h-l-n)}\right.$$

$$\left.\left.\left.-R^{-(g+j+h-l+n)}]-\gamma^{-1}[R^{-(g+j+h-l-n+1)}-R^{-(g+j+h-l+n-1)}]\right)\right\}\right]$$

$$-\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-N+1)}$$

$$+\frac{\xi}{s}\left[\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+2}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-N+2)}\right.$$

$$-\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+1}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-N+1)}$$

$$\left.+\frac{1}{\lambda}\left[\sum_{i=0}^{n-2}\gamma^{-2(n-i-1)}(2\lambda)^{2i-n+2}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-n+2)}-3[\sum_{i=0}^{n-1}\gamma^{-2(n-i)}(2\lambda)^{2i-n+1}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-n+1)}]\right]\right] \quad [27]$$

$$\bar{P}_{m,n}(s)=\gamma^n\lambda^{-1}\left[2\mu\gamma^{-1}\left\{[R^{(n+1)}-R^{-(n+1)}]-\gamma^{-1}[R^n-R^{-n}]\right\}+(s+\lambda+\xi)\left\{[R^n-R^{-n}]-\gamma^{-1}[R^{n-1}-R^{-(n-1)}]\right\}\right]$$

$$\left[\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{p=0}^{h}\sum_{q=0}^{j-l}\sum_{r=0}^{\infty}(-1)^{i+l+h+p+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+k}{k}\binom{j+i+1}{l}\binom{l}{h}\binom{h}{p}\binom{j-l}{p}\binom{j+l+q+r}{r}\right.$$

$$\left.\lambda^{m-i-l+p}2^{m-i-l+p-q}\mu^{l}\gamma^{-(2m+j-i-q)}(s+2\mu+\xi)^{-(m-i-l+p)}(s+\lambda+\xi)^{-(m-i-l+p)}[R^{-(g+j+i+h+p+1)}-R^{-(g+j+i+h+p+2)}]\right]$$

$$-\xi s^{-2}\left[\sum_{i=0}^{N-2}\gamma^{-2(i+1)}(2\lambda)^{2i-N+3}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-N+3)}-\lambda^{-1}\left[\begin{array}{l}\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2\lambda)^{2i-n+1}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-n+1)}\\ -\sum_{i=0}^{n-2}\gamma^{2(n-i-1)}(2\lambda)^{2i-n+2}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-n+2)}\end{array}\right]\right]$$

$$-\xi\lambda^{-1}s^{-1}\left[\sum_{i=0}^{N-1}\gamma^{-2i}(2\lambda)^{2i-N+2}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-N+2)}+\sum_{i=0}^{n-1}\gamma^{2(n-i)}(2\lambda)^{2i-n+1}[(s+2\mu+\lambda+\xi)+\nu]^{-(2i-n+1)}\right] \quad [28]$$

Where

$R=\alpha_1\gamma,\ \nu=\sqrt{(s+2\mu+\lambda+\xi)^2-8\lambda\mu}$ *and* $g=2[N(l+k+h)+h]$

We are now in a position to complete the solution for the joint distribution of X (t) and Y (t). Taking the

Inverse Laplace transform of [27] and [28], using the tables [3, 8], we have:

$$P_{0,n}(t)=\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}2^{-(j+1)}\lambda^{-(j-l+2)}\gamma^{n-2(j+k-l)}\Big[\Big\{2\mu\gamma^{-1}[I_{(g+j+h-l-n-1)}-I_{(g+j+h-l+n+1)}]$$

$$-\gamma^{-1}[I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}]\Big\}e^{-(\lambda+2\mu+\xi)t}+\int_0^t\frac{(t-w)^{(k-l-1)}}{(k-l-1)!}\Big[\Big\{[I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}]-\gamma^{-1}[I_{(g+j+h-l-n+1)}-I_{(g+j+h-l+n-1)}]\Big\}$$

$$(2\sqrt{2\lambda\mu}w)e^{-(\lambda+2\mu+\xi)w}dw\Big]-\sum_{i=0}^{N-1}\gamma^{-(N-1)}(2i-N+1)I_{(2i-N+1)}t^{-1}e^{-(\lambda+2\mu+\xi)t}+\xi\left[\int_0^t\left[[\sum_{i=0}^{N-2}\gamma^{-N}(2i-N+2)I_{(2i-N+2)}]\right.\right.$$

$$\left.\left.-[\sum_{i=0}^{N-1}\gamma^{-(N-1)}(2i-N+1)I_{(2i-N+1)}]+\lambda^{-1}\left[[\sum_{i=0}^{n-2}\gamma^{n}(2i-n+2)I_{(2i-n+2)}]-3[\sum_{i=0}^{n-1}\gamma^{n+1}(2i-n+1)I_{(2i-n+1)}]\right]\right](2\sqrt{2\lambda\mu}w)w^{-1}e^{-(\lambda+2\mu+\xi)w}dw\right] \quad [29]$$

$$P_{m,n}(t)=\left[\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{p=0}^{h}\sum_{q=0}^{j-l}\sum_{r=0}^{\infty}(-1)^{i+l+h+p+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+k}{k}\binom{j+i+1}{l}\binom{l}{h}\binom{h}{p}\binom{j-l}{p}\binom{j+l+q+r}{r}\right.$$

$$\lambda^{(m-i-l+p-1)}2^{(m-i-l+p-q)}\mu^{l}\gamma^{-(2m+j-i-q-n)}\int_0^t\left[\int_0^w\frac{(w-u)^{m-i+k-l-1}}{(m-i+k-l-1)!}\Big[\Big\{2\mu\gamma^{-1}(A-n)I_{(A-n)}+2\mu(A+n+3)I_{(A+n+3)}\right.$$

$$+(A-n+3)I_{(A-n+3)}+\gamma^{-1}(A+n)I_{(A+n)}+\gamma^{-1}[1-2\mu\gamma^{-2}](A+n+2)I_{(A+n+2)}-\gamma(A-n+2)I_{(A-n+2)}$$

$$+2\mu(A-n+1)I_{(A-n+1)}\Big\}(2\sqrt{2\lambda\mu}u)u^{-1}e^{-(\lambda+2\mu+\xi)u}du+\int_0^w\frac{(w-u)^{m-i+k-l-2}}{(m-i+k-l-2)!}\Big[(A-n+1)I_{(A-n+1)}-2(A+n+1)I_{(A+n+1)}$$

$$\left.+(A-n+3)I_{(A-n+3)}+\gamma^{-1}(A+n)I_{(A+n)}+\gamma[(A+n+2)I_{(A+n+2)}-(1+\gamma^{-2})(A-n+2)I_{(A-n+2)}]\Big](2\sqrt{2\lambda\mu}u)u^{-1}e^{-(\lambda+2\mu+\xi)u}du\right]dw$$

$$-\xi\int_0^t(t-w)\left[\sum_{i=0}^{N-2}\gamma^{-(N-1)}(2i-N+3)I_{(2i-N+3)}-\lambda^{-1}[\sum_{i=0}^{n-1}\gamma^{n+1}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^{n}(2i-n+2)I_{(2i-n+2)}]\right](2\sqrt{2\lambda\mu}w)w^{-1}e^{-(\lambda+2\mu+\xi)w}dw$$

$$-\xi\lambda^{-1}\int_0^t\left[\sum_{i=0}^{N-1}\gamma^{-(N-2)}(2i-N+2)I_{(2i-N+2)}+\sum_{i=0}^{n-1}\gamma^{n+1}(2i-n+1)I_{(2i-n+1)}\right](2\sqrt{2\lambda\mu}w)w^{-1}e^{-(\lambda+2\mu+\xi)w}dw \quad [30]$$

Where

$$A=g+j+i+p, I_v(\alpha t)\equiv I_v \text{ and } \alpha=2\sqrt{2\lambda\mu}$$

## Number of Times the System Reaches its Capacity in Time t under Catastrophic Effects

Setting y=1 in [18], we have:

$$\bar{H}(x,1;s)=(s+\xi)^{-1}\left[1+\xi\,\bar{P}_{.}(x,s)-\lambda(1-x)\,\bar{P}_{N-1}(x,s)\right] \quad [31]$$

Using [21] in [31], we have:

$$\bar{H}(x,1;s)=(s+\xi)^{-1}\left[1+\xi\bar{P}_{.}(x,s)-\frac{\lambda(1-x)[(s+\xi)(s+2\lambda)+2\mu\xi]V(1)}{\left[\begin{array}{l}\lambda^2 s[x(s+2\mu+\xi)^{-1}\{\lambda T(N+1)+(s+\xi)T(N)-(s+\lambda+\xi)T(N-1)\}\\+(1-x)\{2V(1)+T(N)+\lambda^{-1}(s+\lambda+\xi)T(N-1)\}\end{array}\right]}\right] \quad [32]$$

Setting x=0 in [32], we have:

$$\bar{P}_{0,.}(s)=(s+\xi)^{-1}\left[1-\frac{[(s+\xi)(s+2\lambda)+2\mu\xi]V(1)}{s[2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)]}\right] \quad [33]$$

Differentiating [32], m times w.r.t. x and dividing both sides by m! and setting x=0, we have:

$$\bar{P}_{m,.}(s)=\left[\frac{\begin{array}{l}\lambda^2V(1)[(s+\xi)(s+2\lambda)+2\mu\xi][\lambda^2T(N+1)+\lambda(s+\xi)T(N)-\lambda(s+\lambda+\xi)T(N-1)-(s+2\mu+\xi)\\\{2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)\}]^{m-1}\end{array}}{s^2(s+2\mu+\xi)^{m-1}[2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)]^{m+1}}\right], m\geq 1 \quad [34]$$

Expanding the right hand sides of [33] and [34], we have:

$$\bar{P}_{0,\cdot}(s)=s^{-1}\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j-k}\sum_{h=0}^{k}\sum_{k=0}^{j}\sum_{r=0}^{\infty}(-1)^{k+l+h}\binom{j}{k}\binom{j-k}{l}\binom{k}{h}\binom{j+r-1}{r}2^{-(j+2)}\lambda^{-(k+2)}\gamma^{-2(j+k+1)}(s+\lambda+\xi)^{-(k-l)}\right.$$
$$\left[2\mu\gamma^{-1}[R^{-(g+j-1)}-R^{-(g+j+1)}]-\gamma^{-1}[R^{-(g+j+r-1)}-R^{-(g+j+r+1)}]\right]-\mu\xi s^{-1}\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j-k}\sum_{h=0}^{k}\sum_{k=0}^{j}\sum_{r=0}^{\infty}(-1)^{k+l+h}\binom{j+1}{k}\right.$$
$$\left.\binom{j-k}{l}\binom{k}{h}\binom{j+r}{r}2^{-(j+2)}\lambda^{-(k+1)}\gamma^{-2(j+k+2)}(s+\lambda+\xi)^{-(k-l-1)}[R^{-(g+j+r+1)}-R^{-(g+j+r+2)}]\right]$$

[35]

$$\bar{P}_{m,\cdot}(s)=\left[\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{p=0}^{m-i-1}\sum_{q=0}^{j-k}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{i+l+p+q+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+k}{k}\binom{j+i+1}{l}\binom{m-i-1}{p}\right.$$
$$\binom{l}{h}\binom{j-k}{q}\binom{j+i+l+r}{r}2^{m-i-l+p-q}\lambda^{m-i-l+p-q-1}\gamma^{-(2m+j-i-q-1)}\mu^{l+p}(s+\lambda+\xi)^{-(m-i-k-l-1)}(s+2\mu+\xi)^{-(m-i-l+p+1)}$$
$$\left[[R^{-(B+p+N)}-R^{-(B+p+1)}]-\gamma^{-1}[R^{-(B+p+N+1)}-R^{-(B+p+N+3)}]+\mu\xi(1+\xi)s^{-2}[R^{-(B+r+1)}-R^{-(B+r+2)}]\right]$$

[36]

Where

B = (g + j + i)

Taking the Inverse Laplace transforms of [35] and [36], we have

$$P_{0,\cdot}(t)=\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j-k}\sum_{h=0}^{k}\sum_{k=0}^{j}\sum_{r=0}^{\infty}(-1)^{k+l+h}\binom{j}{k}\binom{j-k}{l}\binom{k}{h}\binom{j+r-1}{r}2^{-(j+2)}\lambda^{-(k+2)}\gamma^{-2(j+k+1)}\left[\int_0^t\frac{(t-w)^{k-l-1}}{(k-l-1)!}\right.\right.$$
$$\left[\int_0^w\left[\gamma^{-1}\{2\mu[(g+j-1)I_{(g+j-1)}-(g+j+1)I_{(g+j+1)}]-[(g+j+r-1)I_{(g+j+r-1)}-(g+j+r+1)I_{(g+j+r+1)}]\right]\right.$$
$$\left.(2\sqrt{2\lambda\mu}u)u^{-1}e^{-(\lambda+2\mu+\xi)u}du\right]dw-\mu\xi\left[\sum_{j=0}^{\infty}\sum_{l=0}^{j-k}\sum_{h=0}^{k}\sum_{k=0}^{j}\sum_{r=0}^{\infty}(-1)^{k+l+h}\binom{j+1}{k}\binom{j-k}{l}\binom{k}{h}\binom{j+r}{r}2^{-(j+2)}\right.$$
$$\lambda^{-(k+1)}\gamma^{-2(j+k+2)}\left[\int_0^t\frac{(t-w)^{k-l-2}}{(k-l-2)!}\left[\int_0^w\left[(g+j+r+1)I_{(g+j+r+1)}-(g+j+r+2)I_{(g+j+r+2)}\right]\right.\right.$$
$$\left.(2\sqrt{2\lambda\mu}u)u^{-1}e^{-(\lambda+2\mu+\xi)u}du\right]dw$$

[37]

$$P_{m,\cdot}(t)=\left[\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{k=0}^{\infty}\sum_{p=0}^{m-i-1}\sum_{q=0}^{j-k}\sum_{h=0}^{l}\sum_{r=0}^{\infty}(-1)^{i+l+p+q+h}\binom{m-1}{i}\binom{m+j}{j}\binom{j+k}{k}\binom{j+i+1}{l}\right.$$
$$\binom{m-i-1}{p}\binom{l}{h}\binom{j-k}{q}\binom{j+i+l+r}{r}2^{m-i-l+p-q}\lambda^{m-i-l+p-q-1}\gamma^{-(2m+j-i-q-1)}\mu^{l+p}\left[\int_0^t\frac{(t-w)^{m-i-l+p-2}}{(m-i-l+p-2)!}\right.$$

$$\left[\int_0^w \frac{(w-u)^{m-i-k-l-2}}{(m-i-k-l-2)!}\Big[[(B+h+p+N)I_{(B+h+p+N)}-(B+h+p+1)I_{(B+h+p+1)}]-\gamma[(B+h+p+N+1)\right.$$

$$I_{(B+h+p+N+1)}-(B+h+p+N+3)I_{(B+h+p+N+3)}]\Big](2\sqrt{2\lambda\mu}u)u^{-1}e^{-(\lambda+2\mu+\xi)u}+\mu\xi(1+\xi)\left[\int_0^u (u-v)\right.$$

$$\left.\left.\Big[[(B+h+r+1)I_{(B+h+r+1)}-(B+h+r+2)I_{(B+h+r+2)}]\Big](2\sqrt{2\lambda\mu}v)v^{-1}e^{-(\lambda+2\mu+\xi)v}dv\right]du\right]dw \quad [38]$$

## Measure of Effectiveness

Two measures of effectiveness of immediate interest are the expectation $\{\mu_{X(t)}(N)\}$ and the variance $\{\sigma^2_{X(t)}(N)\}$ of the distribution of the number of times the system reaches its capacity in time interval (0, t].

**Expectation $\{\mu_{X(t)}(N)\}$**

$$\bar{\mu}_{X(s)}(N)=\left[\frac{d}{dx}\bar{H}(x,1;s)\right]_{x=1} \quad [39]$$

Using [32] in [39] and after some algebraic calculation, we have:

$$\bar{\mu}_{X(s)}(N)=\left[\sum_{j=0}^{\infty}\sum_{i=0}^{k}\sum_{l=0}^{j+k+1}\sum_{p=0}^{l}\sum_{k=0}^{\infty}(-1)^{i+l}\binom{j+k}{j}\binom{j+k+l}{l}\binom{l}{p}\binom{k}{i}\gamma^{-2(j-i-k)+k(N+1)}(s+\lambda+\xi)^{-(j-i-k)}\right.$$

$$(s+2\mu+\xi)^{-(i-l+p-1)}\left[\lambda[R^{-(H+1)}-R^{-(H-1)}]+2\lambda\xi(\lambda+\mu)s^{-1}[R^{-(H+1)}-R^{-(H+3)}]\right] \quad [40]$$

Taking the inverse Laplace transform, we have:

$$\mu_{X(t)}(N)=\left[\sum_{j=0}^{\infty}\sum_{i=0}^{k}\sum_{l=0}^{j+k+1}\sum_{p=0}^{l}\sum_{k=0}^{\infty}(-1)^{i+l}\binom{j+k}{j}\binom{j+k+l}{l}\binom{l}{p}\binom{k}{i}\gamma^{-2(j-i-k)+k(N+1)}\left[\int_0^t\frac{(t-w)^{j-i-k-1}}{(j-i-k-1)!}\right.\right.$$

$$\left[\int_0^w\frac{(w-u)^{i-l+p-2}}{(i-l+p-2)!}\Big[\lambda[(H+1)I_{(H+1)}-(H-1)I_{(H-1)}]+2\lambda\xi(\lambda+\mu)\left[\int_0^u\Big[[(H+1)I_{(H+1)}-(H+3)I_{(H+3)}]\right.\right.$$

$$\left.\left.\left.(2\sqrt{2\lambda\mu v})v^{-1}e^{-(\lambda+2\mu+\xi)v}dv\right]du\right]dw\right] \quad [41]$$

Where

H= [3(j + i + k + l +1) + N]

**Variance $\{\sigma^2_{X(t)}(N)\}$**

$$\sigma^2_{X(t)}(N)=K(t)+\mu_{X(t)}(N)[1-\mu_{X(t)}(N)] \quad [42]$$

Where

$$K(t) = \sum_{m=2}^{\infty} m(m-1)P_{m,.}(t)$$

Taking Laplace transform of K (t), we have:

$$\bar{K}(s) = \sum_{m=2}^{\infty} m(m-1)\bar{P}_{m,.}(s) \quad \bar{K}(s) = \left[\frac{d^2}{dx^2}\bar{H}(x,1;s)\right]_{x=1} \qquad [43]$$

Using [32] in [43] and after some algebraic calculation, we have

$$\bar{K}(s) = \Bigg[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{p=0}^{\infty}\sum_{l=0}^{j}(-1)^{k+l}\binom{i+1}{i}\binom{j+i+1}{j}\binom{i}{k}\binom{j}{l}\binom{j+i+p+1}{p}\lambda^{-(i+j+2)}\gamma^{-(2k+2l+1+p(N+1))}$$

$$\Big[(s+\xi)^{-(j+2)}(s+\lambda+\xi)^{-i}[2\mu\xi s^{-1}+4\mu(2\mu+\xi)s^{-2}+2\mu\xi(2\mu+\xi)^2 s^{-3}]+(s+\xi)^{-(j+1)}(s+\lambda+\xi)^{-i}$$

$$[2(2\mu+\lambda+\xi)s^{-1}+(2\mu+\xi)(2\mu+4\lambda+\xi)s^{-2}+2\lambda(2\mu+\xi)^2 s^{-3}]\Big]\Big[R^{-(C-1)}-R^{-(C+1)}\Big]-\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}$$

$$\sum_{l=0}^{j}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{k+l}\binom{i+1}{i}\binom{j+i+2}{j}\binom{i}{k}\binom{j}{l}\binom{j+i+p+1}{p}\binom{j+r+1}{r}\lambda^{-(i+j+3)}\gamma^{-(2k+2l+1+p(N+2))}\Big[(s+\xi)^{-(j+3)}$$

$$(s+\lambda+\xi)^{-(i-k)}[2\mu\xi s^{-1}+4\mu(2\mu+\xi)s^{-2}+2\mu\xi(2\mu+\xi)^2 s^{-3}]+(s+\xi)^{-(j+2)}(s+\lambda+\xi)^{-(i-k)}[2(2\mu+\lambda+\xi)s^{-1}$$

$$+(2\mu+\xi)(2\mu+4\lambda+\xi)s^{-2}+2\lambda(2\mu+\xi)^2 s^{-3}]\Big]\Big[R^{-(C+2N)}-R^{-(C+2(N+1))}\Big] \qquad [44]$$

Where

C = [i + j + 2 (k + I + p + 1) (N+1)]

Taking the Inverse Laplace transform, we have

$$\text{K(t)}=\Bigg[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{p=0}^{\infty}\sum_{l=0}^{j}(-1)^{k+l}\binom{i+1}{i}\binom{j+i+1}{j}\binom{i}{k}\binom{j}{l}\binom{j+i+p+1}{p}\lambda^{-(i+j+2)}\gamma^{-(2k+2l+1+p(N+1))}\int_0^t\frac{(t-w)^{j+1}}{(j+1)!}$$

$$\Bigg[\int_0^w\frac{(w-u)^{i-1}}{(i-1)!}\Bigg[[2\mu\xi\int_0^u+4\mu(2\mu+\xi)\int_0^u\frac{(u-v)}{1!}+2\mu\xi(2\mu+\xi)^2\int_0^u\frac{(u-v)^2}{2!}][(C-1)I_{(C-1)}-(C+1)I_{(C+1)}]$$

$$(2\sqrt{2\lambda\mu v})v^{-1}e^{-(\lambda+2\mu+\xi)}dv\Big]du\Big]dw+\int_0^t\frac{(t-w)^j}{j!}\Bigg[\int_0^w\frac{(w-u)^{i-1}}{(i-1)!}\Bigg[[2(2\mu+\lambda+\xi)\int_0^u+(2\mu+\xi)(2\mu+4\lambda+\xi)\int_0^u\frac{(u-v)}{1!}$$

$$+2\lambda(2\mu+\xi)^2\int_0^u\frac{(u-v)^2}{2!}][(C-1)I_{(C-1)}-(C+1)I_{(C+1)}](2\sqrt{2\lambda\mu v})v^{-1}e^{-(\lambda+2\mu+\xi)}dv\Big]du\Big]dw\Big]-\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{k+l}$$

$$\binom{i+1}{i}\binom{j+i+2}{j}\binom{i}{k}\binom{j}{l}\binom{j+i+p+1}{p}\binom{j+r+1}{r}\lambda^{-(i+j+3)}\gamma^{-(2k+2l+1+p(N+2))}\Bigg[\int_0^t\frac{(t-w)^{j+2}}{(j+2)!}\Bigg[\int_0^w\frac{(w-u)^{i-k-1}}{(i-k-1)!}[2\mu\xi$$

$$\int_0^u +4\mu(2\mu+\xi)\int_0^u \frac{(u-v)}{1!}+2\mu\xi(2\mu+\xi)^2\int_0^u \frac{(u-v)^2}{2!}][(C+2N)I_{(C+2N)}-(C+2N+2)I_{(C+2N+2)}](2\sqrt{2\lambda\mu v})v^{-1}e^{-(\lambda+2\mu+\xi)}dv\Big]$$

$$du\Big]dw+\int_0^t \frac{(t-w)^{j+1}}{(j+1)!}\Bigg[\int_0^w \frac{(w-u)^{i-k-1}}{(i-k-1)!}\Bigg[[2(2\mu+\lambda+\xi)\int_0^u +(2\mu+\xi)(2\mu+4\lambda+\xi)\int_0^u \frac{(u-v)}{1!}+2\lambda(2\mu+\xi)^2\int_0^u \frac{(u-v)^2}{2!}]$$

$$[(C+2N)I_{(C+2N)}-(C+2N+2)I_{(C+2N+2)}](2\sqrt{2\lambda\mu v})v^{-1}e^{-(\lambda+2\mu+\xi)}dv\Big]du\Big]dw\Big] \quad [45]$$

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