

Boundary Value Problem of Second Order Ordinary Differential Inclusion

D.S. Palimkar

*Department of Mathematics,
Vasantrao Naik College, Nanded, M.S., India
E-mail: dspalimkar@rediffmail.com*

Abstract

In this paper, prove the existence theorem for the boundary value problem of second order multi-valued convex ordinary differential inclusion.

Keywords: random differential inclusion, multi-valued random operator, random solution, random boundary value problem etc.

2000 Mathematics Subject Classifications: 60H25, 47H10

Introduction

Let R denote the real line and let $\mathcal{P}_f(R)$ denote the class of all non-empty subsets of R with a property f . In particular, $\mathcal{P}_{cl}(R)$, $\mathcal{P}_{bd}(R)$, $\mathcal{P}_{cv}(R)$, and $\mathcal{P}_{cp}(R)$ denote respectively the classes of closed, bounded, convex and compact subsets of R . Similarly $\mathcal{P}_{cl,bd}(R)$ and $\mathcal{P}_{cp,cv}(R)$ denote respectively the classes of all closed-bounded and compact-convex subsets of R . Let $J = [t_0, t_1]$ be a closed and bounded interval in R for some real numbers $t_0, t_1 \in R$ with $t_0 < t_1$. Now consider the two point boundary value problem (in short BVP) of second order differential inclusions

$$-x''(t) \in F(t, x(t), x'(t)) \text{ a.e. } t \in J \quad (1.1)$$

satisfying the boundary conditions

$$\left. \begin{aligned} a_0 x(t_0) - a_1 x'(t_1) &= c_0 \\ b_0 x(t_0) + b_1 x'(t_1) &= c_1 \end{aligned} \right\} \quad (1.2)$$

where the functions involved in (1.1) and (1.2) satisfy the following properties:

$$F : J \times R \times R \rightarrow \mathcal{P}_f(R),$$

- a. $a_0, a_1, b_0, b_1 \in R^+$
- b. satisfying $a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ and
- c. $c_0, c_1 \in R$.

By a solution of the BVP (1.1) – (1.2) we mean a function $x \in AC^1(J, R)$ whose second derivative exists and is a member of $L^1(J, R)$ in $F(t, x, x')$, i.e. there exists a $v \in L^1(J, R)$ such that $v(t) \in F(t, x(t), x'(t))$ for a.e. $t \in J$ and $-x''(t) = v(t)$ for all $t \in J$ satisfying (1.2), where $AC^1(J, R)$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J .

The special cases of the BVP (1.1)-(1.2) have been discussed in the literature for existence of the solutions. The special case of the form

$$-x''(t) = f(t, x(t), x'(t)), \text{ a.e. } t \in J \quad (1.3)$$

Satisfying the boundary conditions (1.2) where $f : J \times R \rightarrow R$, $a_0, a_1, b_0, b_1 \in R_+$, $c_0, c_1 \in R$ and $a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0$ has been discussed in Bernfeld and Lakshmikantham [2] for the existence of solutions. Again when $c_0 = c_1, a_1 = 0 = b_1, a_0 = b_0$, and F not depending on x' , the BVP (1.1) – (1.2) reduces to

$$y'' \in F(t, y) \text{ a.e. } t \in J, \quad y(t_0) = y(t_1). \quad (1.4)$$

where $y = -x$. This is a BVP of second order differential inclusions considered in Benchohra and Ntouyas[3]. Finally, the special case of the BVP consisting of the equation

$$-y''(t) \in F(t, y(t)), \text{ a.e. } t \in J \quad (1.5)$$

satisfying the boundary conditions (1.2) has been studied in Halidias and Papageorgiou[6] via the method of lower and upper solutions. Thus the BVP (1.1) - (1.2) is more general and so is its importance in the theory of differential inclusions. Here in the present chapter, we discuss the BVP (1.1) - (1.2) via a nonlinear alternative of Leray-Schauder type and on a selection theorem for lower semi continuous maps.. In this paper, prove the main existence results for the BVP (1.1) – (1.2) when the right hand side has convex.

Preliminaries

Theorem 2.1: (Covitz and Nadler[5]) Let X be a complete metric space and let $T : X \rightarrow \mathcal{P}_{cl}(X)$ be a multi-valued contraction. Then the fixed point set $\mathcal{F}(T)$ of T is non-empty and closed set in X .

Theorem 2.2: (O'Regan[8]) Let U and \bar{U} be the open and closed subsets in a normed linear space X such that $0 \in U$ and let $T : \bar{U} \rightarrow \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued operator. Then either

- i. the operator inclusion $x \in Tx$ has a solution, or
- ii. there is an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$, where ∂U is the boundary of U .

Corollary 2.1: Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_{cp,cv}(X)$ be a completely continuous multi-valued operator. Then either

- i. the operator inclusion $x \in Tx$ has a solution, or
- ii. there is an element $u \in X$ such that $\|u\| = r$ satisfying $\lambda u \in Tu$ for some $\lambda > 1$.

Corollary 2.2: Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be the open and closed balls in a normed linear space X centered at origin 0 of radius r and let $T : \overline{\mathcal{B}_r(0)} \rightarrow X$ be a completely continuous multi-valued map. Then either

- i. the operator inclusion $x = Tx$ has a solution, or
- ii. there is an element $u \in X$ such that $\|u\| = r$ and $u = \lambda Tu$ for some $\lambda < 1$.

Let $F : J \times R \rightarrow \mathcal{P}_{cp}(R)$ be a multi-valued function. We assign to F , a multi-valued operator $S_F^1 : C(J, R) \rightarrow \mathcal{P}_f(L^1(J, R))$ defined by

$$S_F^1(x) = \left\{ v \in L^1(J, R) \mid v(t) \in F(t, x(t), x'(t)) \quad a.e. \quad t \in J \right\}.$$

The multi-valued operator S_F^1 is called Nemytskii or selection operator associated with the multi-function F .

Now we state a selection theorem due to Bressan and Colombo[4].

Theorem 2.3: Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}_f(L^1(J, R))$ be a multi-valued operator which has property (BC) . Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \rightarrow L^1(J, R)$

such that $g(y) \in N(y)$ for every $y \in Y$.

Existence Results

Lemma[9] 3.1: If $f \in L^1(J, R)$, then the BVP

$$-x''(t) = f(t) \quad a.e. \quad t \in J \quad \text{and} \quad \begin{cases} a_0x(t_0) - a_1x'(t_1) = c_0 \\ b_0x(t_0) + b_1x'(t_1) = c_1 \end{cases} \quad (3.1)$$

has a unique solution x given by

$$x(t) = z(t) + \int_{t_0}^{t_1} G(t, s) f(s) ds, \quad t \in J, \quad (3.2)$$

where z is a unique solution of the homogeneous differential equation

$$-x''(t) = 0 \quad a.e. \quad t \in J \quad \text{and} \quad \begin{cases} a_0x(t_0) - a_1x'(t_1) = c_0 \\ b_0x(t_0) + b_1x'(t_1) = c_1 \end{cases} \quad (3.3)$$

given by

$$z(t) = \frac{c_0a_1(t_1 - t) + c_0b_1 + c_1a_0(t - t_0) + c_1b_0}{a_0a_1(t_1 - t_0) + a_0b_1 + a_1b_0}, \quad t \in J, \quad (3.4)$$

and $G(t, s)$ is the Green's function associated to the differential equation

$$-x''(t) = 0 \quad a.e. \quad t \in J \quad \text{and} \quad \begin{cases} a_0x(t_0) - a_1x'(t_1) = 0 \\ b_0x(t_0) + b_1x'(t_1) = 0 \end{cases} \quad (3.5)$$

given by

$$G(t, s) = \begin{cases} \frac{(a_1(t_1 - t) + b_1)(a_0(s - t_0) + b_0)}{a_0a_1(t_1 - t_0) + a_0b_1 + a_1b_0}, & t_0 \leq s \leq t \leq t_1, \\ \frac{(a_1(t_1 - s) + b_1)(a_0(t - t_0) + b_0)}{a_0a_1(t_1 - t_0) + a_0b_1 + a_1b_0}, & t_0 \leq t \leq s \leq t_1, \end{cases} \quad (3.6)$$

Remark 3.1: It is known that the function z belongs to the class $C^1(J, R)$. Therefore it is bounded on J and there is a constant $C_1 > 0$ with

$$C_1 = \max \left\{ \frac{c_0a_1(t_1 - t_0) + c_0b_1 + c_1a_0(t_1 - t_0) + c_1b_0}{a_0a_1(t_1 - t_0) + a_0b_1 + a_1b_0}, \frac{c_0b_1 - c_0a_1 + c_1a_0 + c_1b_0}{a_0a_1(t_1 - t_0) + a_0b_1 + a_1b_0} \right\}$$

such that $\|z\| = \max \left\{ \sup_{t \in J} |z(t)|, \sup_{t \in J} |z'(t)| \right\} \leq C_1$.

Remark 3.2: It is easy to see that the Green's function $G(t, s)$ of Lemma 3.1 is continuous in $J \times J$ and $G_t(t, s)$ is continuous in $(a, b) \times (a, b) \setminus \{(t, t) | t \in J\}$ and satisfy the inequalities

$$|G(t, s)| = G(t, s) \leq \frac{(a_1(t_1 - t_0) + b_1)(a_0(t_1 - t_0) + b_0)}{a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0} = K_1, \quad (3.7)$$

and

$$\begin{aligned} |G_t(t, s)| &= \begin{cases} \frac{|-a_1|(a_0(s - t_0) + b_0)}{a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0}, & t_0 < s < t < t_1, \\ \frac{(a_1(t_1 - s) + b_1)a_0}{a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0}, & t_0 < t < s < t_1. \end{cases} \\ &= \max \left\{ \frac{a_1(a_0(t_1 - t_0) + b_0)}{a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0}, \frac{(a_1(t_1 - t_0) + b_1)a_0}{a_0 a_1(t_1 - t_0) + a_0 b_1 + a_1 b_0} \right\} \\ &= K_2. \end{aligned} \quad (3.8)$$

Definition 3.1: A multi-valued map $F : J \times R \times R \rightarrow \mathcal{P}_f(R)$ is called Caratheodory if

- i. $t \mapsto F(t, x, y)$ is measurable for all $x, y \in R$, and
- ii. $(x, y) \mapsto F(t, x, y)$ is upper semi-continuous for almost all $t \in J$.

Further a Caratheodory multi-valued function F on $J \times R \times R$ is called L^1 -Caratheodory if

- iii. for each real number $k > 0$, there exists a function $h_k \in L^1(J, R)$ such that

$$\|F(t, x, y)\|_{\mathcal{P}} = \sup \{|v| : v \in F(t, x, y)\} \leq h_k(t), \quad a.e. \ t \in J$$

for all $x, y \in \mathbb{R}$ with $|x| \leq k, |y| \leq k$.

Then we have the following lemmas due to Lasota Opial.

Lemma 3.1: If $\dim(X) < \infty$ and $F : J \times X \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ L^1 -Caratheodory, then $S_F^1(x) \neq \emptyset$ for each $x \in X$.

Lemma 3.2: Let X a Banach space, F an L^1 -Caratheodory multi-valued map with $S_F^1 \neq \emptyset$ and $\mathcal{L} : L^1(J, X) \rightarrow C(J, X)$ be a linear continuous mapping. Then the operator

$$\mathcal{L} \circ S_F^1 : C(J, X) \rightarrow \mathcal{P}_{cp,cv}(C(J, X))$$

has a closed graph in $C(J, X) \times C(J, X)$.

We list here the following assumptions:

- (H_1) The multi $F(t, x, y)$ has compact and convex values for each $(t, x, y) \in J \times R \times R$.
- (H_2) F is Caratheodroy.
- (H_3) There exists a function $\phi \in L^1(J, R)$ with $\phi(t) > 0$ for a.e. $t \in J$ and there is a nondecreasing function $\psi : R^+ \rightarrow (0, \infty)$ such that

$$\|F(t, x, y)\|_{\mathcal{P}} = \sup \{ \|u\| : u \in F(t, x, y) \} \leq \phi(t)\psi(\max\{|x|, |y|\})$$

for a.e. $t \in J$ and for all $x, y \in R$.

Theorem 6.3: 5 Assume that $(H_1) - (H_3)$ hold. Suppose that there is a real number $r > 0$ such that, $r > C_1 + \max\{K_1, K_2\} \|\phi\|_{L^1} \psi(r)$, (3.9)

where C_1, K_1 and K_2 are the constants defined in remark 3.2. Then the BVP (1.1) – (1.2) has at least one solution u such that $\|u\| \leq r$.

Proof: Let $X : AC^1(J, R)$ and consider an open ball $\mathcal{B}_r(0)$ centered at origin of radius r , where r satisfies the condition given in (3.9). The problem of existence of solution of BVP (1.1) – (1.2) reduces to finding the solution of the integral inclusion

$$x(t) \in z(t) + \int_{t_0}^{t_1} G(t, s) F(s, x(s), x'(s)) ds, \quad t \in J. \tag{3.10}$$

Define a multi-valued map $T : \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_f(AC^1(J, R))$ by

$$Tx = \left\{ u \in AC^1(J, R) : u(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds, \quad v \in \overline{S_F^1(x)} \right\}. \tag{3.11}$$

We shall show that the multi T satisfies all the conditions of Corollary 2.2. The proof will be given in several steps.

Step I: We prove that Tx is a convex subset of $AC^1(J, R)$ for each $x \in AC^1(J, R)$. Let $u_1, u_2 \in Tx$. Then there exist v_1 and v_2 in $S_F^1(x)$ such that

$$u_j(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v_j(s)ds, \quad j = 1, 2.$$

Since $F(t, x, y)$ has convex values for all $x, y \in R$, one has for $0 \leq k \leq 1$

$$[kv_1 + (1-k)v_2](t) \in S_F^1(x)(t), \quad \forall t \in J.$$

As a result we have

$$[ku_1 + (1-k)u_2](t) = z(t) + \int_{t_0}^{t_1} G(t, s)[kv_1(s) + (1-k)v_2(s)]ds.$$

Therefore $[ku_1 + (1-k)u_2] \in Tx$ and consequently T has convex values in $AC^1(J, R)$.

Step II: T maps bounded sets in to bounded sets in $AC^1(J, R)$. To see this, let B be a bounded set in $AC^1(J, R)$. Then there exists a real number $q > 0$ such that $\|x\| \leq q, \forall x \in B$.

Now for each $u \in Tx$, there exists a $v \in S_F^1(x)$ such that

$$u(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds.$$

Then for each $t \in J$,

$$\begin{aligned} |u(t)| &\leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)||v(s)|ds \\ &\leq |z(t)| + \int_{t_0}^{t_1} |G(t, s)|\phi(s)\psi(\max\{|x(t)|, |x'(t)|\})ds. \end{aligned}$$

Again,

$$\begin{aligned}
|u'(t)| &\leq |z'(t)| + \int_{t_0}^{t_1} |G_t(t, s)| |v(s)| ds \\
&\leq |z'(t)| + \int_{t_0}^{t_1} |G_t(t, s)| \phi(s) \psi \left(\max \{|x(t)|, |x'(t)|\} \right) ds.
\end{aligned}$$

This further implies that

$$\begin{aligned}
\|u\| &= \max_{t \in J} \{|x(t)|, |x'(t)|\} \\
&= \max_{t \in J} \max \{|z(t)|, |z'(t)|\} \\
&\quad + \int_{t_0}^{t_1} \max_{t, s \in J} \{|G(t, s)|, |G_t(t, s)|\} \phi(s) \psi \left(\max \{|x(s)|, |x'(s)|\} \right) ds \\
&\leq C_1 + \max \{K_1, K_2\} \|\phi\|_{L^1} \psi(q)
\end{aligned}$$

for all $u \in Tx \subset \cup T(B)$. Hence $\cup T(B)$ is bounded.

Step III: Next we show that T maps bounded sets into equi-continuous sets. Let B be a bounded set as in step II, and $u \in Tx$ for some $x \in B$. Then there exists $v \in S_F^1(x)$

such that, $u(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v(s)ds$.

Then for any $t, \tau \in J$, we have

$$\begin{aligned}
|u(t) - u(\tau)| &\leq |z(t) - z(\tau)| + \left| \int_{t_1}^{t_2} G(t, s)v(s)ds - \int_{t_0}^{t_2} G(\tau, s)v(s)ds \right| \\
&\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t, s) - G(\tau, s)| |v(s)| ds \\
&\leq |z(t) - z(\tau)| + \int_{t_0}^{t_1} |G(t, s) - G(\tau, s)| \phi(s) \psi(q) ds.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|u'(t) - u'(\tau)| &\leq |z'(t) - z'(\tau)| + \int_{t_0}^{t_1} |G_1(t, s) - G_1(\tau, s)| \phi(s) \psi(q) ds \\
&\leq \int_{t_0}^{\tau} \phi(s) \psi(q) ds.
\end{aligned}$$

Therefore from the above two estimates, it follows that

$$\max \left\{ |u(t) - u(\tau)|, |u'(t) - u'(\tau)| \right\} \rightarrow 0, \quad \text{as } t \rightarrow \tau.$$

As a result $\cup T(B)$ is an equi-continuous set in $AC^1(J, R)$. Now an application of Arzela-Ascoli theorem yields that the multi T is completely continuous operator on $AC^1(J, R)$.

Step IV: Next we prove that T has closed graph. Let $\{x_n\} \subset AC^1(J, R)$ be a sequence such that $x_n \rightarrow x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Tx_n$, for each $n \in \mathbb{N}$ such that $y_n \rightarrow y_*$. We must show that $y_* \in Tx_*$. Since $y_n \in Tx_n$, there exists a $v_n \in S_F^1(x_n)$ such that

$$y_n(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v_n(s)ds.$$

Consider the linear and continuous operator $\mathcal{L} : L^1(J, R) \rightarrow AC^1(J, R)$ defined by

$$\mathcal{L}v(t) = \int_{t_0}^{t_1} G(t, s)v(s)ds.$$

$$\begin{aligned} \text{Now } \max_{t \in J} \left\{ |y_n(t) - z(t) - (y_*(t) - z(t))|, |y_n'(t) - z'(t) - (y_*'(t) - z'(t))| \right\} \\ = \|y_n - y_*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

From Lemma 3.2 it follows that $(\mathcal{K} \circ S_F^1)$ is a closed graph operator and from the definition of \mathcal{L} one has

$$y_n - z \in (\mathcal{L} \circ \overline{S_F^1}(x_n)).$$

As $x_n \rightarrow x_*$ and $y_n \rightarrow y_*$, there is a $v_* \in S_F^1(x_*)$ such that

$$y_*(t) = z(t) + \int_{t_0}^{t_1} G(t, s)v_*(s)ds.$$

Hence the multi T is an upper semi-continuous operator on $\overline{\mathcal{B}_r(0)}$.

Thus, T is an upper semi-continuous and compact operator on $\overline{\mathcal{B}_r(0)}$. Now an application of Corollary 2.2 yields that either (i) the operator inclusion $x \in Tx$ has a solution in $\overline{\mathcal{B}_r(0)}$, or (ii) there is an element $u \in X$ with $\|u\| = r$ such that $\lambda u \in Tu$ for some $\lambda > 1$. We show that the assertion (ii) is not possible. Assume the

contrary. Then proceeding with the arguments as in Step II, we obtain which is a contradiction to (3.9). Hence BVP (1.1) – (1.2) has a solution u on J such that $\|u\| \leq r$.

References

- [1] J. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, 1984.
- [2] S. Bernfield and V. Lakshmikantham, *An Introduction to Boundary Value Problem*, Academic Press, New York, 1974.
- [3] M. Benchora and S. K. Ntouyas, On second order differential inclusions with periodic boundary conditions, *Acta. Math. Univ. Comeniana* LXIX (2000), 173-181.
- [4] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* 90, 1988, 69-86.
- [5] H. Covitz and S. B. Nadler, Multi-valued contraction mappings in generalized metric spaces, *Israel J. Math.* 8, 1970, 5-11.
- [6] N. Halidias and N. S. Papageorgiou, Second order multi-valued boundary value problems, *Arch. Math (Brno)* 34, 1998, 267-284.
- [7] Hu and N. S. Papageorgiou, *Hand Book of Multi-valued Analysis Vol.-I, Theory* Kluwer Academic Publisher, Dordrecht, Boston, London, 1997
- [8] D. O'Regan, Fixed point theory for closed multifunctions, *Arch.Math.(Brno)* 34, 1998, 191-197.