

## The Value of the Games using Fuzzy Nash Equilibrium

D. Stephen Dinagar<sup>1</sup> and T. Porchelvi<sup>2</sup>

<sup>1</sup>*PG& Research Department of Mathematics,  
T.B.M.L. College, Porayar-609 307, Tamil Nadu, India*

<sup>2</sup>*PG Department of Mathematics, Poompuhar College,  
Melaiyur-609 107, Tamil Nadu, India  
E-mail: selvi4685@yahoo.com*

### Abstract

In this paper the method of determining the value of the game in fuzzy environment, using trapezoidal fuzzy numbers is proposed. A new approach called fuzzy Nash equilibrium which provides a way for obtaining the optimum strategies for  $m \times n$  fuzzy payoff matrix is employed. Relevant definitions and numerical examples are also given.

**Keywords:** Fuzzy sets, Fuzzy numbers, Fuzzy Pay-off matrix, Trapezoidal fuzzy numbers.

### Introduction

Game theory is the study of conflict situations wherein the decisions made by a person will depend not only upon his own actions but also upon the actions of his opponents. According to Martin Shubik[4], "Game theory is a method of the study of decision making in situations of conflict. It deals with human processes in which the individual decision-unit is not in complete control of other decision-units entering into the environment".

In classical game theory [5], precise estimates are assumed to be available for the payoffs in the matrix and this is a major limitation. In reality, it is highly difficult to arrive at such precise estimates. The aim of this work is to obviate this serious problem.

Let us suppose that estimates are represented by fuzzy numbers for the payoffs. Fuzzy numbers enable one to describe the uncertainty about the actual value of a numerical variable. In [7], some methods for finding value of fuzzy games are explained. In [8], the same is discussed for Constrained fuzzy games, an extension of

fuzzy games. This approach will prove to be very realistic and fuzzy numbers are helpful in decision making for specifying acceptable or unacceptable value of a numeric variable.

## Preliminaries

### Definition

A trapezoidal fuzzy number  $\tilde{A} = (a_1, a_2, a_3, a_4)$  is defined by the membership function as

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a_1) / (a_2 - a_1) & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ (x - a_4) / (a_3 - a_4) & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{Otherwise} \end{cases}$$

### Order Relation on Fuzzy Numbers

Let  $\tilde{a} \approx (a_1, a_2, a_3, a_4)$  and  $\tilde{b} \approx (b_1, b_2, b_3, b_4)$  be any two symmetric trapezoidal fuzzy numbers. Let us define  $\tilde{a} \preceq \tilde{b}$  if and only if  $a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, a_4 \leq b_4$ . One can verify that  $\preceq$  is a partial order relation. We define  $\tilde{a} \prec \tilde{b}$  (or  $\tilde{b} \prec \tilde{a}$ ) if for at least one  $i, a_i < b_i$ .

### Unit and Zero fuzzy numbers

Unit fuzzy number ( $\tilde{I}$ ) is one whose de-fuzzification is unity (i.e. if  $\tilde{a} \approx (a_1, a_2, a_3, a_4)$  is the unit fuzzy number then  $(a_1 + a_2 + a_3 + a_4) / 4 = 1$ ) and similarly the zero fuzzy number ( $\tilde{O}$ ) is one whose de-fuzzification is zero.

### Equivalent fuzzy numbers

Two fuzzy numbers are said to be equal if their de-fuzzifications are equal i.e., if  $[(a_1 + a_2 + a_3 + a_4) / 4] = [(b_1 + b_2 + b_3 + b_4) / 4]$ . Then We say that  $\tilde{a}$  is equal to  $\tilde{b}$  and write  $\tilde{a} \approx \tilde{b}$ .

### Normalized fuzzy number

A trapezoidal fuzzy number is said to be normalized if the difference between any two of its entries is unity.

### Normalized fuzzy game

A fuzzy game is said to be normalized fuzzy game if all of its entries were normalized fuzzy numbers.

### Operations on Fuzzy Numbers:[7]

For  $\tilde{a} \approx (a_1, a_2, a_3, a_4)$  and  $\tilde{b} \approx (b_1, b_2, b_3, b_4)$  the following operations are defined.

**Addition:**

$$\tilde{a} \oplus \tilde{b} \approx (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4)$$

**Subtraction:**

$$\tilde{a} \ominus \tilde{b} \approx (a_1-b_4, a_2-b_3, a_3-b_2, a_4-b_1)$$

**Multiplication:**

$$\begin{aligned} \tilde{a} \otimes \tilde{b} &\approx (a_1(b_1+b_2+b_3+b_4)/4, a_2(b_1+b_2+b_3+b_4)/4, a_3(b_1+b_2+b_3+b_4)/ \\ &\quad 4, a_4(b_1+b_2+b_3+b_4)/4), \text{if } \tilde{b} \succeq \tilde{0} \text{ and } \tilde{a} \succeq \tilde{0} \\ \tilde{b} \otimes \tilde{a} &\approx (a_4(b_1+b_2+b_3+b_4)/4, a_3(b_1+b_2+b_3+b_4)/4, a_2(b_1+b_2+b_3+b_4)/ \\ &\quad 4, a_1(b_1+b_2+b_3+b_4)/4), \text{if } \tilde{b} \preceq \tilde{0} \end{aligned}$$

**Division:**

$$\begin{aligned} \tilde{a} \oslash \tilde{b} &\approx (4a_1/(b_1+b_2+b_3+b_4), 4a_2/(b_1+b_2+b_3+b_4), \\ &\quad 4a_3/(b_1+b_2+b_3+b_4), 4a_4/(b_1+b_2+b_3+b_4)), \text{if } \tilde{b} \succeq \tilde{0} \\ \tilde{a} \oslash \tilde{b} &\approx (4a_4/(b_1+b_2+b_3+b_4), 4a_3/(b_1+b_2+b_3+b_4), \\ &\quad 4a_2/(b_1+b_2+b_3+b_4), 4a_1/(b_1+b_2+b_3+b_4)), \text{if } \tilde{b} \preceq \tilde{0} \end{aligned}$$

where  $\tilde{b}$  is not the zero fuzzy number.

**Scalar Multiplication:**

If  $k \neq 0$  is a scalar,  $k \tilde{a}$  is defined as,

$$k \tilde{a} \begin{cases} (ka_1, ka_2, ka_3, ka_4) & \text{if } k > 0 \\ \approx \\ (ka_4, ka_3, ka_2, ka_1) & \text{if } k < 0 \end{cases}$$

**Note:**

$$\begin{aligned} \tilde{a} \oplus \tilde{b} &\approx \tilde{b} \oplus \tilde{a} \\ \tilde{a} \ominus \tilde{b} &\approx (-1)(\tilde{b} \ominus \tilde{a}) \\ \tilde{a} \otimes \tilde{b} &\approx \tilde{b} \otimes \tilde{a} \end{aligned}$$

Any fuzzy number can be written as a normalized fuzzy number which is equal to it. For example, (-3,1,4, 6) is replaced by (1/2,3/2,5/2,7/2)

**Fuzzy Matrix Games:**

Let the fuzzy strategy spaces for player I and player II be,

$$\begin{aligned} \tilde{S}_1 &= \left\{ \tilde{X} \approx (\tilde{x}_1, \dots, \tilde{x}_m)^T: \tilde{x}_i \succeq \tilde{0} \text{ for all } i, \sum_{i=1}^m \tilde{x}_i \approx \tilde{I} \right\} \\ \tilde{S}_2 &= \left\{ \tilde{Y} \approx (\tilde{y}_1, \dots, \tilde{y}_n)^T: \tilde{y}_i \succeq \tilde{0} \text{ for all } i, \sum_{i=1}^n \tilde{y}_i \approx \tilde{I} \right\} \end{aligned} \quad j=1$$

respectively and expected payoff is given by,

$$\tilde{K}(\tilde{X}, \tilde{Y}) \approx \sum_{i=1}^m \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_i \otimes \tilde{y}_j \approx \tilde{X}^T \otimes \tilde{A} \otimes \tilde{Y}$$

where  $\tilde{A} \approx (\tilde{a}_{ij})_{m \times n}$  is a given fuzzy matrix, called the fuzzy payoff matrix, and  $\tilde{K}: \tilde{S}_1 \times \tilde{S}_2 \rightarrow F(\mathcal{R})$ . Then the triplet  $\{\tilde{S}_1, \tilde{S}_2, \tilde{A}\}$  is called the fuzzy game and is denoted by  $\mathcal{FMG} \approx \{\tilde{S}_1, \tilde{S}_2, \tilde{A}\}$ .

**Fuzzy value and fuzzy saddle**

Let  $\tilde{A} \approx (\tilde{a}_{ij})_{m \times n}$  is the given fuzzy payoff matrix . Then the fuzzy matrix game is said to have a fuzzy saddle point if

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} \tilde{a}_{ij} \approx \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} \tilde{a}_{ji}$$

and the common value is taken to be the fuzzy value ( $\tilde{V}$ ) of the game. Otherwise the  $\mathcal{FMG}$  is said to have no fuzzy saddle.

**Illustration**

Consider the normalized fuzzy game,

$$\begin{pmatrix} (1, 2, 3, 4) & (6/7, 13/7, 20/7, 27/7) \\ (1/8, 9/8, 7/8, 25/8) & (12/7, 19/7, 26/7, 33/7) \end{pmatrix}$$

The row minima are respectively  $(6/7, 13/7, 20/7, 27/7)$ ,  $(1/8, 9/8, 17/8, 25/8)$  and their maximum is  $(6/7, 13/7, 20/7, 27/7)$  while the column maxima are  $(1, 2, 3, 4)$ ,  $(12/7, 19/7, 26/7, 33/7)$ , and the minimum being  $(1, 2, 3, 4)$ . Thus, there is no saddle point for this game. In such a situation one has to consider the adoption of mixed strategy.

**Fuzzy Nash Equilibrium**

**Definition**

Fuzzy Nash equilibrium is a solution concept of a fuzzy game involving two or more players, in which each player is assumed to know the fuzzy equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy unilaterally. If each player has chosen a fuzzy strategy and no player can benefit by changing his or her fuzzy strategy while the other players keep theirs unchanged, then the current set of fuzzy strategy choices and the corresponding payoffs constitute a fuzzy Nash equilibrium.

**Theorem**

Consider the fuzzy matrix game,  $\mathcal{FMG} \approx (\tilde{S}_1, \tilde{S}_2, \tilde{A})$  where ,

$$\tilde{S}_1 \approx \left\{ \tilde{X} \approx (\tilde{x}_1, \dots, \tilde{x}_m)^T \in F^m(\mathcal{R})_{\oplus} \text{ and } \sum_{i=1}^m \tilde{x}_i \approx \tilde{I} \right\} \quad i=1$$

$$\tilde{S}_2 \approx \left\{ \tilde{Y} \approx \{(\tilde{y}_1, \dots, \tilde{y}_n)^T \in \mathcal{F}^n(\mathcal{R})_{\oplus} \text{ and } \sum_{j=1}^n \tilde{y}_j \approx \tilde{I} \} \right. \quad j=1$$

$\tilde{A} \approx (\tilde{a}_{ij})_{m \times n}$ . Then the fuzzy value of the FMG is obtained by solving the LPP given by,

$$\begin{aligned} & \text{Max } \tilde{y}_1' \oplus \dots \oplus \tilde{y}_n' \\ & \text{Subject to,} \\ & \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j' \leq \tilde{I} \quad (i = 1, \dots, m) \\ & \tilde{y}_j' \geq \tilde{O} \quad (j = 1, \dots, n) \end{aligned}$$

**Proof**

From the definition of fuzzy nash equilibrium we have,

$$\tilde{V} \approx \min_{\tilde{Y} \in \tilde{S}_2} \max_{\tilde{X} \in \tilde{S}_1} \tilde{X}^T \otimes \tilde{A} \otimes \tilde{Y} \approx \max_{\tilde{X} \in \tilde{S}_1} \min_{\tilde{Y} \in \tilde{S}_2} \tilde{X} \otimes \tilde{A} \otimes \tilde{Y}$$

Then Player II chooses his fuzzy strategy  $\tilde{Y} \in \tilde{S}_2$  that minimizes,

$$\max_{\tilde{X} \in \tilde{S}_1} \sum_{i=1}^m \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_i \otimes \tilde{y}_j \text{ and}$$

$$\max_{\tilde{X} \in \tilde{S}_1} \sum_{i=1}^m \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{x}_i \otimes \tilde{y}_j \approx \max_{1 \leq i \leq m} \sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j \quad (\approx \tilde{z} \text{ say})$$

Then we have,

$$\tilde{V} \approx \min \tilde{z}$$

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j \leq \tilde{z} \quad (i = 1, \dots, m)$$

$$\begin{aligned} & \sum_{j=1}^n \tilde{y}_j \approx \tilde{I} \\ & \tilde{y}_j' \geq \tilde{O} \quad (j = 1, \dots, n) \quad (\because \tilde{Y} \in \tilde{S}_2) \end{aligned}$$

By taking  $\tilde{y}_j' \approx \tilde{y}_j \otimes \tilde{z}$  we have,

$$\begin{aligned} & \text{max } \tilde{y}_1' \oplus \dots \oplus \tilde{y}_n' \\ & \text{subject to,} \end{aligned}$$

$$\sum_{j=1}^n \tilde{a}_{ij} \otimes \tilde{y}_j' \leq \tilde{I} \quad (i = 1, \dots, m)$$

$$\tilde{y}_j' \geq \tilde{O} \quad (j = 1, \dots, n)$$

**Note**

To find the fuzzy strategy for maximizing player, we take the dual of the above fuzzy LPP. Or, it can be found by using the above procedure to solve a modified fuzzy payoff matrix which is the transpose and negation of  $\tilde{A}$

**Example**

Consider the normalized fuzzy game,

$$\left[ \begin{array}{cc} (1,2,3,4) & 1/7(6,13,20,27) \\ 1/8(1,9,17,25) & 1/7(12,19,26,33) \end{array} \right]$$

Then the fuzzy LPP becomes,

$$\text{Max } \tilde{z} = \tilde{y}_1 \oplus \tilde{y}_2$$

$$\tilde{y}_1(1,2,3,4) \oplus \tilde{y}_2 1/7(6,13,20,27) \leq (-1/2, 1/2, 3/2, 5/2)$$

$$\tilde{y}_1 1/8(1,9,17,25) \oplus \tilde{y}_2 1/7(12,19,26,33) \leq (-1/2, 1/2, 3/2, 5/2)$$

$$\tilde{y}_1, \tilde{y}_2 \geq \tilde{O}$$

From table 1 and 2 we have

$$\tilde{I} / \tilde{V} = 1/40(-29, 1, 31, 61)$$

$$\tilde{V} = (-1/2, 1/2, 3/2, 5/2) / [1/40(-29, 1, 31, 61)]$$

$$= 5/2(-1/2, 1/2, 3/2, 5/2)$$

Under simplex method the fuzzy value of the FMG in the normalized form is ,

$$\tilde{V} \approx (-5/4, 5/4, 15/4, 25/4)$$

**Table 1**

| $C_B$                    | $Y_B$ | $X_B$                   | $y_1$                      | $y_2$                       | $s_1$                    | $s_2$                    |
|--------------------------|-------|-------------------------|----------------------------|-----------------------------|--------------------------|--------------------------|
| $(-3/2, -1/2, 1/2, 3/2)$ | $s_1$ | $(-1/2, 1/2, 3/2, 5/2)$ | $(1, 2, 3, 4)$             | $1/7(6, 13, 20, 27)$        | $(-1/2, 1/2, 3/2, 5/2)$  | $(-3/2, -1/2, 1/2, 3/2)$ |
| $(-3/2, -1/2, 1/2, 3/2)$ | $s_2$ | $(-1/2, 1/2, 3/2, 5/2)$ | $1/8(1, 9, 17, 25)$        | $1/7(12, 19, 26, 33)$       | $(-3/2, -1/2, 1/2, 3/2)$ | $(-1/2, 1/2, 3/2, 5/2)$  |
|                          |       | $(-3, -1, 1, 3)$        | $1/16(-139, -57, 25, 107)$ | $1/28(-304, -120, 64, 248)$ | $(-3, -1, 1, 3)$         | $(-3, -1, 1, 3)$         |

**Table 2**

| <b>C<sub>B</sub></b>   | <b>Y<sub>B</sub></b> | <b>X<sub>B</sub></b>      | <b>Y<sub>1</sub></b> | <b>Y<sub>2</sub></b>           | <b>S<sub>1</sub></b>      | <b>S<sub>2</sub></b>       |
|------------------------|----------------------|---------------------------|----------------------|--------------------------------|---------------------------|----------------------------|
| (-1/2, 1/2, 3/2, 5/2)  | y <sub>1</sub>       | 4/10(-1/2, 1/2, 3/2, 5/2) | 4/10(1, 2, 3, 4)     | 4/70(6, 13, 20, 27)            | 4/10(-1/2, 1/2, 3/2, 5/2) | 4/10(-3/2, -1/2, 1/2, 3/2) |
| (-3/2, -1/2, 1/2, 3/2) | s <sub>2</sub>       | 1/20(-35, -7, 21, 49)     | 1/8(-24, -8, 8, 24)  | 1/280(-345, 199, 743, 1287)    | 1/20(-55, -27, 1, 29)     | (-1/2, 1/2, 3/2, 5/2)      |
|                        |                      | 1/40(-29, 1, 31, 61)      | (-3, -1, 1, 3)       | 1/560(-1677, -207, 1263, 2733) | 1/20(-112, -32, 48, 128)  | (-3, -1, 1, 3)             |

**Conclusion**

In this work a new method to find out the value of the games in fuzzy environment is proposed and fuzzy estimations are represented by trapezoidal fuzzy numbers. To find the value of the games in fuzzy environment, an important notion like fuzzy Nash Equilibrium is used and the trapezoidal fuzzy arithmetic operations are also employed. Some other notions and approaches may also be used to solve the fuzzy problems in game theory in future.

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