A Partial Solution to an Open Problem in Strict Menger Probabilistic Metric Spaces

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Abstract

In this paper, we give partial solution to the open problem 2.10 posed in [3].

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Introduction

In Sastry et.al [3] an open problem in a strict Menger space is given, incidentally observing a fallacy in the argument of a result of Servet Kutucku and Sushil Sharma [6].

We use the notion of strict Menger space given in [3] and make use of this to prove a fixed point theorem (Theorem 2.1) in strict Menger spaces with min t-norm. An open problem (open problem 2.2) is also given at the end of the paper.

We start with

Definition 1.1: [4] A function \( F: \mathbb{R} \rightarrow [0,1] \) is called a distribution function if \( F \) is non-decreasing, left continuous and \( \inf_{x \in \mathbb{R}} F(x) = 0 \) and \( \sup_{x \in \mathbb{R}} F(x) = 1 \).

Definition 1.2: [4] A triangular norm \( *: [0,1] \times [0,1] \rightarrow [0,1] \) is a function satisfying the following conditions:
1. \( \alpha \ast 1 = \alpha \forall \alpha \in [0,1] \),
2. \( \alpha \ast \beta = \beta \ast \alpha \forall \alpha, \beta \in [0,1] \),
3. \( \gamma \ast \delta \geq \alpha \ast \beta \forall \alpha, \beta, \gamma, \delta \in [0,1] \) with \( \gamma \geq \alpha \) and \( \delta \geq \beta \),
4. \( (\alpha \ast \beta) \ast \gamma = \alpha \ast (\beta \ast \gamma) \forall \alpha, \beta, \gamma \in [0,1] \).

A triangular norm is also denoted by t-norm. For any \( a, b \in [0,1] \), if we define \( a \ast b = \min \{a, b\} \), then \( \ast \) is a t-norm and is denoted by ‘min’.

**Definition 1.3:** [4] Let \( X \) be a non-empty set and let \( F: X \times X \rightarrow \mathcal{D} \) (The set of distribution functions). For \( p, q \in X \), we denote the image of the pair \( (p, q) \) by \( F_{p,q} \) which is a distribution function so that \( F_{p,q}(x) \in [0,1] \), for every real \( x \). Suppose \( F \) satisfies:

1. \( F_{p,q}(x) = 1 \) for all \( x > 0 \) if and only if \( p = q \),
2. \( F_{p,q}(0) = 0 \),
3. \( F_{p,q}(x) = F_{q,p}(x) \),
4. If \( F_{p,q}(x) = 1 \) and \( F_{q,r}(y) = 1 \) then \( F_{p,r}(x+y) = 1 \) where \( p, q, r \in X \).

Then \( (X, F) \) is called a probabilistic metric space.

**Definition 1.4:** [4] Let \( X \) be a non empty set, \( \ast \) a t-norm and \( F: X \times X \rightarrow \mathcal{D} \) satisfies:

1. \( F_{p,q}(0) = 0 \forall p, q \in X \),
2. \( F_{p,q}(x) = 1 \forall x > 0 \) if and only if \( p = q \),
3. \( F_{p,q}(x) = F_{q,p}(x) \forall p, q \in X \),
4. \( F_{p,r}(x+y) \geq \ast (F_{p,q}(x), F_{q,r}(y)) \forall x, y \geq 0 \) and \( p, q, r \in X \).

Then the triplet \( (X, F, \ast) \) is called a Menger space.

**Definition 1.5:** [5]

1. Let \( (X, F, \ast) \) be a Menger space and \( p \in X \). For \( \varepsilon > 0, 0 < \lambda < 1 \), the \( (\varepsilon, \lambda) \)-neighbourhood of \( p \) is defined as \( U_p(\varepsilon, \lambda) = \{ q \in X: F_{p,q}(\varepsilon) > 1 - \lambda \} \). It may be observed that, if \( \ast \) is continuous then the topology induced by the family \( \{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1 \} \) is a Hausdorff topology on \( X \) and is known as the \( (\varepsilon, \lambda) \) - topology.
2. A sequence \( \{x_n\} \) in \( X \) is said to converge to \( p \in X \) in the \( (\varepsilon, \lambda) \) -topology, if for any \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{p,q}(\varepsilon) > 1 - \lambda \) where \( n > N \).
3. A sequence \( \{x_n\} \) in \( X \) is said to be a Cauchy sequence in the \( (\varepsilon, \lambda) \)-topology, if for \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) there exists a positive integer \( N = N(\varepsilon, \lambda) \) such that \( F_{x_m, x_n}(\varepsilon) > 1 - \lambda \) for all \( m, n > N \).
4. A Menger space \( (X, F, \ast) \), where \( \ast \) is continuous, is said to be complete if
every Cauchy sequence in \( X \) is convergent in the \((\varepsilon, \lambda)\)-topology.

**Definition 1.6:** [1] Let \( \ast \) be a t-norm. For any \( a \in [0,1] \), write \( \ast_0 (a) = 1 \) and \( \ast_1 (a) = \ast (\ast_0 (a), a) = \ast (1, a) = a \)

In general define \( \ast_{n+1} (a) = \ast (\ast_n (a), a) \) for \( n = 0, 1, 2 \ldots \)

If \( \ast_n \) is equicontinuous at 1, that is given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( x > 1 - \delta \) implies \( \ast_n (x) > 1 - \varepsilon \) \( \forall n \in \mathbb{N} \).

then we say that \( \ast \) is a Hadzic type t-norm.

We observe that ‘min’ t-norm is of Hadzic type.

**Definitions 1.7:** [6] Two self mappings \( A \) and \( B \) of a Menger space \( (\mathcal{C}, \mathcal{F}, \mathcal{X}) \) are said to be

1. compatible of type (P) if \( F_{ABx_n, BBx_n} (t) \to 1 \) and \( F_{BAX_n, AAX_n} (t) \to 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

2. compatible of type (P1) if \( F_{ABx_n, BBx_n} (t) \to 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

3. compatible of type (P2) if \( F_{BAX_n, AAX_n} (t) \to 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ax_n, Bx_n \to z \) for some \( z \) in \( X \) as \( n \to \infty \).

We need the following lemma.

**Lemma 1.8:** [2] Let \( (\mathcal{C}, \mathcal{F}, \mathcal{X}) \) be a Menger space with Hadzic-type t-norm \( \ast \) and \( 0 < a < 1 \). Suppose \( \{x_n\} \) is a sequence in \( X \) such that for any \( s > 0 \), \( F_{x_n, x_{n+1}} (s) \geq F_{x_0, x_1} (\frac{s}{a^n}) \).

Then \( \{x_n\} \) is a Cauchy sequence.

**Definition 1.9:** [3] Let \( (\mathcal{C}, \mathcal{F}, \mathcal{X}) \) be a Menger space such that \( F_{x,y} (t) \) is strictly increasing in \( t \) whenever \( x \neq y \). Then \( (\mathcal{C}, \mathcal{F}, \mathcal{X}) \) is called a strict Menger space.

**Example 1.10:** [3] Let \( (X, d) \) be a metric space. Define \( F_{x,y} (t) = \frac{t}{t+d(x,y)} \) \( \forall t > 0 \) and \( x, y \in X \). If t-norm \( \ast \) is \( a \ast b = \min \{a,b\} \) \( \forall a, b \in [0,1] \), then \( (X, F, \ast) \) is a strict Menger space.

**Main results**

The following theorem is given in Sastry et al [3].

**Theorem 2.1:** [3] Let \( P, Q, R \) and \( C \) be self maps of a complete strict Menger space \( (X, F, \ast) \) with min t-norm \( \ast \) satisfying:

1. \( P(X) \subseteq R(X), Q(X) \subseteq C(X) \),
2. there exists a constant \( k \in (0,1) \) such that \( F_{P_X, Q_Y} (kt) \geq F_{C_X, R_Y} (t) \) \( F_{P_X, C_X} (t) * F_{Q_Y, R_Y} (t) * F_{P_X, R_Y} (2t) * F_{Q_Y, C_X} (2t) \) for all \( x, y \in X, t > 0 \),
3. either P or C is continuous,
4. the pairs (P, C) and (Q, R) are both compatible of type (P₁) or type (P₂).

Then P, Q, R and C have a unique common fixed point.

The following open problem is posed in [3].

Open Problem 2.2: [3] Is Theorem 2.1 valid if \( t \) in condition (b) is replaced by \( \alpha t \) where \( \alpha \in (1,2) \)?

Now we prove a fixed point theorem for four self maps on a complete strict Menger space which gives a partial solution to the open problem 2.2.

Theorem 2.3: Let P, Q, R and C be self maps of a complete strict Menger space \((X, F, *)\) with min t-norm * satisfying:
1. \( P(X) \subseteq R(X), Q(X) \subseteq C(X) \), there exists a constant \( k \in (0,1) \) and \( \alpha \in (2k, 2) \) such that
   \[ F_{P_X Q_Y}(kt) \geq F_{C_X R_Y}(t) \ast F_{P_X C_X}(t) \ast F_{Q_Y R_Y}(t) \ast F_{P_X C_X}(at) \]
2. either P or C is continuous,
3. the pairs (P, C) and (Q, R) are both compatible of type (P₁) or type (P₂).

Then P, Q, R and C have a unique common fixed point.

Proof: Let \( x_0 \in X \). By (a), there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in X such that
\[ p_{x_n} = R_{x_n} = y_{2n} \] and \( q_{x_n} = C_{x_n} = y_{2n+1} \) for \( n = 0, 1, 2, \ldots \)
Step 1: By taking \( x = x_2n, y = x_{2n+1} \) for all \( t > 0 \) in (b), we get
\[ F_{P_{x_n} Q_{y_n}}(kt) \geq F_{C_{x_n} R_{y_n}}(t) \ast F_{P_{x_n} C_{x_n}}(t) \ast F_{Q_{y_n} R_{y_n}}(t) \ast F_{P_{x_n} C_{x_n}}(at) \]
\[ \Rightarrow F_{y_{2n+1}}(kt) \geq F_{y_{2n-1}}(t) \ast F_{y_{2n-1}}(t) \ast F_{y_{2n+1}}(t) \ast F_{y_{2n}}(at) \]
\[ \geq F_{y_{2n-1}}(t) \ast F_{y_{2n+1}}(t) \ast F_{y_{2n-1}}(t) \ast F_{y_{2n}}(at) \]
\[ \geq F_{y_{2n-1}}(at) \ast F_{y_{2n}}(at) \]
\[ \Rightarrow x \in \text{a Cauchy sequence.} \]
Since \((X, F,*)\) is complete, it converges to a point \( z \) in X. Also its sub sequences \( \{P_{x_n}\} \rightarrow z, \{C_{x_n}\} \rightarrow z, \{Q_{x_n+1}\} \rightarrow z \) and \( \{R_{x_n+1}\} \rightarrow z \).

Case (i): \( C \) is continuous, (P, C) and (Q, R) are compatible of type (P₂)
\[ CC_{x_n} \rightarrow Cz, CP_{x_n} \rightarrow Cz \] (\( \because C \) is continuous)
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and $PPx_{2n} \rightarrow Cz$ (⋄ (P, C) is compatible of type (P2))

By taking $x = Px_{2n}, y = x_{2n+1}$ in (b), we get

$$F_{PPx_{2n}Qx_{2n+1}}(kt) \geq F_{CPx_{2n}Rx_{2n+1}}(t) * F_{PPx_{2n}Cx_{2n}}(t) * F_{Qx_{2n+1}Rx_{2n+1}}(t) * F_{PPx_{2n}Rx_{2n+1}}(at) * F_{Qx_{2n+1}Cx_{2n}}(at)$$

On letting $n \rightarrow \infty$, we get

$$F_{Czz}(kt) \geq F_{Czz}(t) * F_{Czz}(t) * F_{Czz}(t) * F_{Czz}(at) * F_{Czz}(at)$$

If $Cz \neq z$, $F_{Czz}(kt) < F_{Czz}(t)$ (⋄ $0 < k < 1$)

and $F_{Czz}(kt) < F_{Czz}(at)$ (⋄ $k < 2k < \alpha$)

⋄ $F_{Czz}(kt) < F_{Czz}(t) * F_{Czz}(at) \leq F_{Czz}(kt)$, a contradiction.

Hence $Cz = z$.

**Step 3:** By taking $x = z, y = x_{2n+1}$ in (b), we get

$$F_{PzQx_{2n+1}}(kt) \geq F_{CzRx_{2n+1}}(t) * F_{PzCx_{2n}}(t) * F_{Qx_{2n+1}Rx_{2n+1}}(t) * F_{PzRx_{2n+1}}(at) * F_{Qx_{2n+1}Cx_{2n}}(at)$$

On letting $n \rightarrow \infty$, we get

$$F_{Pz,z}(kt) \geq F_{z,z}(t) * F_{z,z}(t) * F_{z,z}(t) * F_{z,z}(at) * F_{z,z}(at)$$

$$\geq F_{Pz,z}(t) * F_{Pz,z}(at)$$

Thus $Pz = z$.

**Step 4:** Since $P(X) \subseteq R(X)$, there exists $w \in X$ such that $z = Pz = Rw$

By taking $x = x_{2n}, y = w$ in (b), we get

$$F_{Px_{2n}Qw}(kt) \geq F_{CzRx_{2n}}(t) * F_{PzCx_{2n}}(t) * F_{QwRx_{2n}}(t) * F_{PzRx_{2n}}(at) * F_{QwCx_{2n}}(at)$$

On letting $n \rightarrow \infty$, we get

$$F_{z,w}(kt) \geq F_{z,w}(t) * F_{z,w}(t) * F_{Qwz}(t) * F_{z,w}(at) * F_{Qwz}(at)$$

$$\geq F_{Qwz}(t) * F_{Qwz}(at)$$

Thus $Qw = w$.

⋄ $Rw = Qw = z$

Since $(Q, R)$ is compatible of type (P2), we have $RQw = QQw$.

Therefore $Rz = Qz$.

**Step 5:** By taking $x = x_{2n}, y = z$ in (b), we get

$$F_{PzQz}(kt) \geq F_{CzRx_{2n}}(t) * F_{PzCx_{2n}}(t) * F_{QzRz}(t) * F_{PzRx_{2n}}(at) * F_{QzCx_{2n}}(at)$$

On letting $n \rightarrow \infty$, we get

$$F_{z,z}(kt) \geq F_{z,z}(t) * F_{z,z}(t) * F_{QzQz}(t) * F_{z,z}(at) * F_{QzQz}(at)$$

$$\geq F_{QzQz}(t) * F_{QzQz}(at)$$

Thus $Qz = z$.

⋄ $Pz = Qz = Cz = Rz = z$. 
i.e. $z$ is a common fixed point for $P$, $Q$, $R$ and $C$.

**Case (ii):** $P$ is continuous and $(P, C), (Q, R)$ are both compatible of type $(P_2)$

$PQx_{2n} \to Pz, PCx_{2n} \to Pz \ (\because P$ is continuous)

$CPx_{2n} \to Pz \ (\because (P, C)$ is compatible of type $(P_2)$)

**Step 6:** By taking $x = P_{x2n}, y = y_{2n+1}$ in (b), we get

$$F_{PPx_{2n},Qx_{2n+1}}(kt) \geq F_{CPx_{2n},R_{x2n+1}}(t) \ast F_{PPx_{2n},C_{x2n}}(t) \ast F_{Qx_{2n+1},R_{x2n+1}}(at)$$

$$F_{Qx_{2n+1},C_{x2n}}(at)$$

On letting $n \to \infty$, we get

$$F_{Pz,z}(kt) \geq F_{Pz,z}(t) \ast F_{Pz,at} \ast F_{z,z}(at) \ast F_{z,z}(at)$$

$$\geq F_{Pz,z}(t) \ast F_{Pz,at}$$

Thus $Pz = z$.

Using step 4 and step 5, we get $z = Qz = Rz$.

**Step 7:** Since $Q(X) \subseteq C(X)$, there exists $w \in X$ such that $z = Qz = Cw$.

By taking $x = w, y = x_{2n+1}$ in (b), we get

$$F_{Pw,Qx_{2n+1}}(kt) \geq F_{Cw,R_{x2n+1}}(t) \ast F_{Pw,C_{x2n+1}}(t) \ast F_{Qx_{2n+1},R_{x2n+1}}(at) \ast$$

$$F_{Qx_{2n+1},C_{x2n}}(at)$$

On letting $n \to \infty$, we get

$$F_{Pw,z}(kt) \geq F_{z,z}(t) \ast F_{Pw,z}(t) \ast F_{z,z}(at) \ast F_{z,z}(at)$$

$$\geq F_{Pw,z}(t) \ast F_{Pw,at}$$

Thus $z = Pw$, since $z = Qz = Cw$, hence $Pw = Cw$.

$(P, C)$ is compatible of type $(P_2)$, we have $CPw = PPw \ i.e. Cz = Pz$.

$\therefore z = Pz = Cz = Qz = Rz$

i.e. $z$ is a common fixed point for $P, Q, R$ and $C$.

$\therefore z$ is a common fixed point for $P, Q, R$ and $C$ when $C$ is continuous( or $P$ is continuous) and $(P, C), (Q, R)$ are compatible of type $(P_2)$ ( or $(P_1)$).

**Step 8:** For uniqueness $v$ be common fixed point for $P, Q, R$ and $C$.

Take $x = z, y = v$ in the condition (b), we get

$$F_{Pz,Qv}(kt) \geq F_{Cz,Rv}(t) \ast F_{Pz,Cv}(t) \ast F_{Qv,Rv}(t) \ast F_{Pz,Rv}(at) \ast F_{Qv,Cv}(at)$$

$$\Rightarrow F_{z,v}(kt) \geq F_{z,v}(t) \ast F_{z,v}(t) \ast F_{z,v}(at) \ast F_{z,v}(at)$$

$$\geq F_{z,v}(t) \ast F_{z,v}(at)$$

Thus $v = z$.

We conclude our paper with an open problem.

Open Problem 2.4: If $1 > k > \frac{1}{2}$ and $\alpha \in (2k, 2)$, is the result valid?
References


