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An Algorithm for Capacitated n-Index Transportation Problem

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Abstract

In this paper an algorithm was introduced for capacitated transportation problem with n-subscripts.

Introduction

Many world famous researchers like Dentzing, Hitchcock, Kantorovich and Savourine et.al have proposed the theoretical and algorithm bases of the classical two-index transportation problem. Most of the developments are based on linear programming techniques.

Heley [3] and Junginger [4] introduced the multi-index transportation problem with out capacities. In fact, the capacities of the transportation paths are mathematically modeled as additional constraints to express very important real needs. Obviously, this involves some theoretical and algorithmically compilations which are often difficult to treat in general context. This justify in part, as most absence of significant studies related to capacitated transportation problem with an index greater than two.

In this paper, we focus our attention to the capacitated transportation problem with n-index.

Statement of the problem

The capacitated n-index transportation problem (CTP) is formulated as follows:

$$\text{min. } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \ldots \sum_{n=1}^{x} c_{ijk\ldots n} x_{ijk\ldots n}$$

s.t.

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\ldots\sum_{n=1}^{x} x_{ijk\ldots n} = \alpha_i \text{ for all } i = 1,2,3,\ldots,m$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\ldots\sum_{n=1}^{x} x_{ijk\ldots n} = \beta_j \text{ for all } j = 1,2,3,\ldots,n$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\ldots\sum_{n=1}^{x} x_{ijk\ldots n} = \chi_k \text{ for all } k = 1,2,3,\ldots,p$$

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$$\sum_{i=1}^{m}\sum_{j=1}^{n}\ldots\sum_{n-1=1}^{w} x_{ijk\ldots n} = \sigma_n \text{ for all } n = 1,2,3,\ldots,x$$

In this problem $\alpha_i, \beta_j, \chi_k, \ldots, \sigma_n, d_{ijk\ldots n}$ and $c_{1jk..n}$ are given and are such that for all $i,j,k,l,\ldots n$ we have $\alpha_i > 0$, $\beta_j > 0$, $\chi_k > 0, \ldots, \sigma_n > 0, d_{ijk\ldots n} > 0$ and $c_{ijk\ldots n} > 0$.

This problem can be equivalently formulated as the linear program

min. $[\, c^t x \;:\; Ax = b, 0 \le x \le d \,]$

Where $x, c, d \in \Re^{mnp\ldots x}$, $b \in \Re^{m+n+p+\ldots+x}$ and A is a $(m+n+\ldots+x)$ $(mn\ldots x)$ matrix.

A feasible solution $x = x_{ijk\ldots n}$ of (CTP) is called a program.

A program x is called basic if the columns of the sub matrix A_x obtained from A by keeping only the columns corresponding to the variables $x_{ijk\ldots n}$ such that

$0 < x_{ijk\ldots n} < d_{ijk\ldots n}$

are linearly independent.

A basic program x is said to be degenerate if

rank (A_x) = rank (A)

Given a basic program x, the n-tuple ($i,j,\ldots,n$) is called interesting if

$0 < x_{ijk\ldots n} < d_{ijk\ldots n}$

We assume that the following feasibility assumption holds

$$\sum_{i=1}^{m}\alpha_i = \sum_{j=1}^{n}\beta_j = \sum_{k=1}^{p}\chi_k = \ldots = \sum_{n=1}^{x}\sigma_n = H$$

It results that

Rank (A) = (m + n +… + x) – (n-1)

It is useful to present the data of the problem via the following transportation table. It consists of an array of (m + n + …+x) rows and ($mn…x$) columns, n-1 additional rows and an additional columns. The entries of column $P_{ij…n}$ of the first, second, …,n-1 additional rows are for the data of the quantities $d_{ijk…n}$, $c_{ijk…n}$,…,$x_{ijk…n}$ respectively. The additional columns are for the data of quantities $\alpha_i, \beta_j, \chi_k, ..., \sigma_n$ respectively. Finally the entry of the array on the line corresponding to α_i and the column $P_{ij…n}$ is 1 if $i = i'$ and 0 if not. Same as for $\beta_j, \chi_k, ..., \sigma_n$.

Algorithm

The following algorithm shares with the simplex method and the potential methods a structure consisting in two phases, a finite convergence and the use of the pivot principle.

Phase 1: (It finds a basic program or says that (CTP) is not solvable)

Step 1:
Initialization:

For all ($i,j,k,…,n$), $\hat{\alpha}_i = \alpha_i$, $\hat{\beta}_i = \beta_j$, $\hat{\chi}_k = \chi_k$,…, $\hat{\sigma}_n = \sigma_n$ and $b_{ujk...n} = 0$ ($b_{ijk…n}$ is a boolean variable equal to 1 if $x_{ijk…n}$ has already been determined and 0 if not yet),

E = {($i,j,k,…,n$), such that $b_{ijk…n}$ = 0}.

Iteration:

While $E \neq \phi$ do

Choose an n-tuple $(\bar{i}, \bar{j}, \bar{k}, ..., \bar{n}) \in E$, such that $c_{\bar{i}\bar{j}\bar{k}...\bar{n}} = \min_{(i,j,k,...n)\in E} c_{ijk...n}$ take $x_{\bar{i}\bar{j}\bar{k}...\bar{n}} = \min.(\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\chi}_{\bar{k}}, ...\hat{\sigma}_{\bar{n}}, d_{\bar{i}\bar{j}\bar{k}...\bar{n}})$, and $b_{\bar{i}\bar{j}\bar{k}...\bar{n}} = 1$ (i.e., $x_{\bar{i}\bar{j}\bar{k}...\bar{n}}$ is determined), update $\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\chi}_{\bar{k}}, ..., \hat{\sigma}_{\bar{n}}$ as follows $\hat{\alpha}_{\bar{i}} = \alpha_{\bar{i}} - x_{\bar{i}\bar{j}\bar{k}...\bar{n}}$, if $\hat{\alpha}_{\bar{i}} = 0$ then take $x_{\bar{i}jk...n} = 0$ for all $(j,k,...,n) \neq (\bar{j}, \bar{k}, ..., \bar{n})$ and $b_{\bar{i}jk...n} = 1$ for all $(j,k,...,n-1)$.

$\hat{\beta}_{\bar{j}} = \beta_{\bar{j}} - x_{\bar{i}\bar{j}\bar{k}...\bar{n}}$, if $\hat{\beta}_j$ =0 then take $x_{i\bar{j}k...n}$ = 0 for all $(i,k,...,n) \neq (\bar{i}, \bar{k}, ..., \bar{n})$ and $b_{i\bar{j}k...n} = 1$ for all ($i,k,…,n$-1). $\hat{\chi}_k = \chi_{\bar{k}} - x_{\bar{i}\bar{j}\bar{k}...\bar{n}}$,

If $\hat{\chi}_k$ =0 then take $x_{ij\bar{k}...n}$ = 0 for all $(i, j, l, ..., n) \neq (\bar{i}, \bar{j}, \bar{l}, ..., \bar{n})$ and $b_{ij\bar{k}...n} = 1$ for all ($i,j,l,…,n$-1). and similarly for,

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n. $\hat{\sigma}_n = \sigma_{\bar{n}} - x_{\bar{i}\bar{j}\bar{k}...\bar{n}}$, If $\hat{\sigma}_n = 0$ then take $x_{ijk\ldots\bar{n}} = 0$ for all $(j,k,\ldots n-1) \neq (\bar{i},\bar{j},\bar{k},\ldots,\bar{n}-1)$ and $b_{ijk\ldots\bar{n}} = 1$ for all (*i, j, k,…,n-1*).

Step 2:
Take

$$\in = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} c_k = \ldots = \sum_{n=1}^{x} u_n \text{ , such that}$$

$$a_i = \alpha_i - \sum_{j=1}^{n} \sum_{k=1}^{p} \ldots \sum_{n=1}^{x} x_{ijk\ldots n} \text{ with } i = 1,2,\ldots,m$$

$$b_j = \beta_j - \sum_{i=1}^{m} \sum_{k=1}^{p} \ldots \sum_{n=1}^{x} x_{ijk\ldots n} \text{ with } j = 1,2,\ldots,n$$

$$c_k = \chi_k - \sum_{i=1}^{m} \sum_{j=1}^{n} \ldots \sum_{n=1}^{x} x_{ijk\ldots n} \text{ with } k = 1,2,\ldots,p$$

and for

$$u_n = \sigma_n - \sum_{i=1}^{m} \sum_{j=1}^{n} \ldots \sum_{n-1=1}^{w} x_{ijk\ldots n} \text{ with } n = 1,2,\ldots,x$$

i. If $\in = 0$, then $x = (x_{ijk\ldots n})$ is an initial basic program for the problem (CTP), we denote it by $x^{(0)}$. Go to Phase 2.

ii. Construct a problem $CTP(\widetilde{M})$ by the procedure described in (P_1) below, and find an initial basic program $\bar{x}^{(0)}$ for the problem $CTP(\widetilde{M})$,as in step 1.

Then, $x^{(0)}_{m+1,n+1,p+1,\ldots,x+1} = 0$

($\bar{x}^{(0)} = (x_{ijk\ldots n})$, with $i = 1,2,\ldots,m+1$, $j = 1,2,\ldots,n+1$, $k = 1,2,\ldots,p+1,\ldots$, $n = 1,2,\ldots,x+1$).

If $\bar{x}^{(0)}$ is optimal then the problem (CTP) is not solvable. Stop.

Improvement of a basic program for $CTP(\widetilde{M})$.

Initialization: $r = 1, \in > 0$ is given,

Determine $\bar{x}^{(r)}$ as in phase 2.

If $x^{(r)}_{m+1,n+1,\ldots,x+1} = \in$, then $x^{(r)} = (x^{(r)}_{ijk\ldots n})$, with $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $k = 1,2,\ldots,p,\ldots$, $n = 1,2,\ldots,x$, is an initial basic program for the problem (CTP). Go to Phase 2.

If $\bar{x}^{(r)}$ is optimal (Phase 2), then the problem (CTP) is not solvable. Stop.

Do $r = r+1$ and repeat 1), to 3).

Next, we describe the second phase.

Phase 2: (Research of an optimal program for (CTP)

When Phase 2 starts, we known an initial basic program $x^{(0)}$. Take r = 0.

Determine the set $I^{(r)}$ of the interesting n-tuple ($i,j,k,\ldots,n$).

For all $(i, j, k, ..., n) \in I^{(r)}$, solve the linear system

$u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + ... + p_n^{(r)} = c_{ijk\ldots n}$.

For all ($i,j,k,\ldots,n$) $(i, j, k, ..., n) \notin I^{(r)}$ take

$\Delta_{ijk\ldots n}^{(r)} = c_{ijk\ldots n} - (u_i^{(r)} + v_j^{(r)} + ... + p_n^{(r)})$

and

$\Gamma_0^{(r)} = \{\Delta_{ijk\ldots n}^{(r)}$ Such that $x_{ijk\ldots n}^{(r)} = 0\}$.

$\Gamma_d^{(r)} = \{\Delta_{ijk\ldots n}^{(r)}$ Such that $x_{ijk\ldots n}^{(r)} = d_{ujk\ldots n}\}$.

If the following optimality conditions holds

$\Delta_{ijk\ldots n}^{(r)} \geq 0$ For all $\Delta_{ijk\ldots n}^{(r)} \in \Gamma_0^{(r)}$

and

$\Delta_{ijk\ldots n}^{(r)} \leq 0$ For all $\Delta_{ijk\ldots n}^{(r)} \in \Gamma_d^{(r)}$

Then the program $x^{(r)}$ is optimal. Stop.

Determine

$\Delta_{i_0, j_0, k_0, ..., n_0}^{(r)} = \min.[\Delta_{ijk\ldots n}^{0(r)} - \Delta_{ujk\ldots n}^{d(r)}]$ Such that

$\Delta_{i,j,k,\ldots n}^{0(r)} \in \Gamma_0^{(r)}$, with $\Delta_{ijk\ldots n}^{0(r)} < 0$,

$\Delta_{i,j,k,\ldots n}^{d(r)} \in \Gamma_d^{(r)}$, with $\Delta_{ijk\ldots n}^{d(r)} > 0$

and specify if $\Delta_{i_0, j_0, k_0, ..., n_0}^{(r)} \in \Gamma_0^{(r)} (or \in \Gamma_d^{(r)})$.

Construct via the procedure described in (P_2) below, a cycle $\mu^{(r)}$ containing some interesting n-tuple ($i,j,k,\ldots,n$) and the non interesting n-tuple ($i_0,j_0,k_0,\ldots,n_0$) corresponding to $\Delta_{i_0, j_0, k_0, ..., n_0}^{(r)}$.

Take,

$\sigma^{(r)}$ = {($i,j,k,\ldots,n$) such that ($i,j,k,\ldots,n$) is a n-tuple forcoming the cycle $\mu^{(r)}$ },

$\sigma^{(r)-}$ = {($i,j,k,\ldots,n$) such that ($i,j,k,\ldots,n$)$\in \sigma^{(r)}$, with $\alpha_{ijk\ldots n} < 0$ },

$\sigma^{(r)+}$ = {(*i,j,k,…,n*) such that (*i,j,k,…,n*) $\in \sigma^{(r)}$, with $\alpha_{ijk\ldots n} > 0$ },

If $\Delta^{(r)}_{i_0,j_0,k_0,\ldots,n_0} \in \Gamma^{(r)}_0$, determine

$$\theta_1^{(r)} = \min_{(i,j,k,\ldots,n)\in\sigma^{(r)-}} (x^{(r)}_{ijk\ldots n} / -\alpha_{ijk\ldots n}) ,$$

$$\theta_2^{(r)} = \min_{(i,j,k,\ldots,n)\in\sigma^{(r)+}} (d_{ijk\ldots n} - x^{(r)}_{ijk\ldots n} / \alpha_{ijk\ldots n}) ,$$

$$\theta^{(r)} = \min(\theta_1^{(r)}, \theta_2^{(r)}) ,$$

Next, take

$$x^{(r+1)} = \{x^{(r)}_{ijk\ldots n} + \alpha_{ijk\ldots n}\theta^{(r)}, (i,j,k,\ldots,n) \in \sigma^{(r)}\} \mathrm{U} \{x^{(r)}_{ijk\ldots n}, (i,j,k,\ldots,n) \notin \sigma^{(r)}\}.$$

else ($\Delta^{(r)}_{i_0,j_0,k_0,\ldots,n_0} \in \Gamma^{(r)}_d$), determine

$$\theta_1^{(r)} = \min_{(i,j,k,\ldots,n)\in\sigma^{(r)+}} (x^{(r)}_{ijk\ldots n} / \alpha_{ijk\ldots n}) ,$$

$$\theta_2^{(r)} = \min_{(i,j,k,\ldots,n)\in\sigma^{(r)-}} (d_{ijk\ldots n} - x^{(r)}_{ijk\ldots n} / -\alpha_{ijk\ldots n}) ,$$

$$\theta^{(r)} = \min(\theta_1^{(r)}, \theta_2^{(r)})$$

next, take

$$x^{(r+1)} = \{x^{(r)}_{ijk\ldots n} - \alpha_{ijk\ldots n}\theta^{(r)}, (i,j,k,\ldots,n) \in \sigma^{(r)}\} \mathrm{U} \{x^{(r)}_{ijk\ldots n}, (i,j,k,\ldots,n) \notin \sigma^{(r)}\}.$$

Do *r = r+1* and repeat a), to e) until the optimality condition holds.

The above algorithm makes appeal to the following procedures:

(P_1)- Construction of a problem

The problem $CTP(\tilde{M})$ is obtained from problem (CTP) by adding four fictitious points with indices *m+1, n+1, p+1,…, x+1* such that:

$$c_{m+1,n+1,p+1,\ldots,x+1} = 0, c_{m+1,jk\ldots n} = c_{i,n+1,kl\ldots n} = \ldots = c_{ijk\ldots n-1} = \tilde{M}$$

(where $\tilde{M}$ is a very large number) and there are no limitation on the capacities for the paths involving a fictitious point.

(P_2)- Determination of cycles

A cycle $\mu^{(r)}$ is determined by the solving the linear system

$$\sum_{(i,j,k,\ldots,n)\in I(r)} \alpha_{ijk\ldots n} P_{ijk\ldots n} = -P_{i_0 j_0 k_0 \ldots n_0}$$

The non null solutions $\alpha_{ijk\ldots n}$ are called coefficients of the cycle $\mu^{(r)}$.

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