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A New Method of Proportion for Solving Nonlinear Equations

Jayanta Dutta[1] and S.C. Pal[2]

[1]*Dept. of Mathematics, Siliguri Institute of Technology, Siliguri-734009, India.*
[2]*Dept. of Computer Science and Application,*
University of North Bengal, Siliguri-734013, India
[1]*E-mail: mathjd651@yahoo.com*

Abstract

In this paper a new family of efficient iterative methods is presented for solving nonlinear equations based on the proportion of two real parameters, associated with the given equations. In our proposed methods, we generate a single iterative formula and from that single formula it is shown that the methods are linear, quadratic, cubic and higher order of convergence according to the choice of terms from an expression. The methods are supported by various numerical examples and shown that the new proposed methods are effective and comparable to the well known other existing methods.

Keywords: Nonlinear equation, Proportion, Iterative method, Order of convergence.

Introduction

One of the most basic and fundamental problems in numerical analysis is that of finding values of the variable x, which satisfies

$$f(x) = 0 \qquad (1)$$

for a given function f. There are so many existing standard methods are available for solving such equation (1). Of course out of them Newton's method is very popular and useful. In past few years so many authors have considered the nonlinear equations and gave us a new idea towards the solution and compare their result with the well known Newton's methods [1- 4]. Basto et al [1] and Chen at al [2] were presented an iterative formula in which the order of convergence is fixed and in particular, in [1] when the other iterative method is considered by taking more terms in the series solution obtained by Adomian modified method [5, 6] or by considering Taylor's expansion of higher order, the order of convergence was not increased. In [2] exponential regula-falsi iterative method is presented in which the order of convergence is also fixed to second order.

At first we assume that the function f is continuous in [a, b] and posses the n-th order derivatives in (a, b), where a and b are finite real numbers. In our method, we first find out a sufficiently small interval $[a_0, b_0]$ containing the only root α of equation (1) such that $f(a_0)$, $f(b_0)$ are of opposite in signs in [a, b] and further assume that $f'(x)$ has the same sign in $[a_0, b_0]$, so that f(x) is strictly monotone in that interval which contains the only root α.

In this paper we give a new approach to the subject based on the ratio of the two parameters $f(a_0)$ and $f(b_0)$ and the main aim of our paper is that, the order of convergence is increased with the taking of more terms in the single series obtained by the Taylor's expansion. We observed that the ratio of $f(a_0)$ and $f(b_0)$ is very important and $f(a_0)/f(b_0)$ or $f(b_0)/f(a_0)$ is closely associated with the root α and by monitoring the values of $f(a_0)/f(b_0)$ or $f(b_0)/f(a_0)$, we established a new iterative formula (2) to find out the root α. In section (3) we also present a family of iterative methods in which the order of convergence is shown more than third and fourth order according to a choice of a real parameter.

The method

After finding the sufficient small interval $[a_0, b_0]$, the expression for the root α of the equation f(x) = 0 is given by

$$\begin{aligned} \alpha &= b_0 - \frac{b_0 - a_0}{1 - \dfrac{f(a_0)}{f(b_0)}} - D, \text{ if } \left|\frac{f(a_0)}{f(b_0)}\right| < 1 \\ &= a_0 + \frac{b_0 - a_0}{1 - \dfrac{f(b_0)}{f(a_0)}} - D, \text{ if } \left|\frac{f(b_0)}{f(a_0)}\right| < 1 \qquad (2) \\ &= \frac{a_0 + b_0}{2} - D, \text{ if } \left|\frac{f(a_0)}{f(b_0)}\right| = 1 \end{aligned}$$

where D is given by, $D = \dfrac{f(w_0)}{f'(\xi_0)}$, where w_0 is a point in $[a_0, b_0]$ given by

$$w_0 = a_0 - \frac{b_0 - a_0}{f(b_0) - f(a_0)} f(a_0) \qquad (3)$$

which is our well-known Regula-falsi formula and $\min\{\alpha, w_0\} < \xi_0 < \max\{\alpha, w_0\}$ (4) or, D satisfies the expression,

$$D\left[f^{/}(w_0)+\frac{D}{2}f^{//}(w_0)+\frac{D^2}{3!}f^{///}(w_0)+\frac{D^3}{4!}f^{iv}(w_0)+....+\frac{D^{n-2}}{(n-1)!}f^{n-1}(w_0)+\frac{D^{n-1}}{n!}f^{n}(w_0+\theta D)\right]$$

$$= f(w_0),\ \text{where } 0 < \theta < 1 \qquad (5)$$

Here ξ_0 is a point where the tangent to the curve y = f(x) at (ξ_0 , f (ξ_0)) is parallel to the chord joining the points (α, 0) and (w_0, f(w_0)). Since α is unknown we cannot locate exactly the point ξ_0 and consequently, we cannot exactly determine the value of θ. So, we approximate the value of D by taking the first few terms of (5).

A Particular method

If we consider only the first term of the expression (5) then the approximate value of D is given by, $D = \dfrac{f(w_0)}{f^{/}(w_0)}$ and we find the recurrence relation for the (n+1) th approximation of the root α as

$$x_{n+1} = b_n - \frac{b_n - a_n}{1-\dfrac{f(a_n)}{f(b_n)}} - \frac{f(w_n)}{f^{/}(w_n)},\ \text{if } \left|\frac{f(a_n)}{f(b_n)}\right| < 1$$

$$= a_n + \frac{b_n - a_n}{1-\dfrac{f(b_n)}{f(a_n)}} - \frac{f(w_n)}{f^{/}(w_n)},\ \text{if } \left|\frac{f(b_n)}{f(a_n)}\right| < 1 \qquad (6)$$

$$= \frac{a_n + b_n}{2} - \frac{f(w_n)}{f^{/}(w_n)},\ \text{if } \left|\frac{f(a_n)}{f(b_n)}\right| = 1$$

$$\text{where } w_n = a_n - \frac{b_n - a_n}{f(b_n) - f(a_n)} f(a_n),\ a_n \le \alpha \le b_n \qquad (7)$$

Convergence Analysis

In view of (2-4), the true value of α is given by

$$\alpha = b_n - \frac{b_n - a_n}{1-\dfrac{f(a_n)}{f(b_n)}} - \frac{f(w_n)}{f^{/}(\xi_n)},\ \text{if } \left|\frac{f(a_n)}{f(b_n)}\right| < 1$$

$$= a_n + \frac{b_n - a_n}{1-\dfrac{f(b_n)}{f(a_n)}} - \frac{f(w_n)}{f^{/}(w_n)},\ \text{if } \left|\frac{f(b_n)}{f(a_n)}\right| < 1$$

$$= \frac{a_n + b_n}{2} - \frac{f(w_n)}{f^{/}(\xi_n)},\ \text{if } \left|\frac{f(a_n)}{f(b_n)}\right| = 1$$

where $\min\{\alpha, w_n\} < \xi_n < \max\{\alpha, w_n\}$.

Let ε_{n+1} is the error after (n+1)-th iteration.

Hence $\varepsilon_{n+1} = x_{n+1} - \alpha = \dfrac{f^{/}(w_n) - f^{/}(\xi_n)}{f^{/}(w_n) f^{/}(\xi_n)} f(w_n) = \dfrac{(w_n - \xi_n) f^{//}(\eta_n)}{f^{/}(w_n) f^{/}(\xi_n)} f(w_n)$ (8)

where $\min\{w_n, \xi_n\} < \eta_n < \max\{w_n, \xi_n\}$. Now it is not very difficult to show that $w_n \to \alpha$ as $n \to \infty$. Now the expression (7) can be written as

$$w_n = b_n - \frac{b_n - a_n}{f(b_n) - f(a_n)} f(b_n) \tag{9}$$

For simplicity and without loss of generality and in view of (7) and (9), (6) can be written as

$$x_{n+1} = w_n - \frac{f(w_n)}{f^{/}(w_n)}$$

Consider the iteration function g as $g(x) = x - \dfrac{f(x)}{f^{/}(x)}$. Therefore $g^{/}(x) = \dfrac{f^{//}(x) f(x)}{\{f^{/}(x)^2\}}$. Hence the method will be converge if $|g^{/}(x)| < 1$ ie

$$|f(x) f^{//}(x)| < \{f^{/}(x)\}^2, \ x \in [a_0, b_0] \tag{10}$$

Since $x_n = a_n$ or b_n, we have in either case, after considering $\theta_n = w_n - \alpha$,

$$w_n = x_n - \frac{b_n - a_n}{f(b_n) - f(a_n)} f(x_n)$$

or, $w_n - x_n = -\dfrac{b_n - a_n}{(b_n - a_n) f^{/}(\eta^1{}_n)} f(x_n)$, where $a_n < \eta^1{}_n < b_n$

or, $\theta_n - \varepsilon_n = -\dfrac{(x_n - \alpha) f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}$, where $\min\{x_n, \alpha\} < \eta^2{}_n < \max\{x_n, \alpha\}$

or, $\theta_n = \left\{1 - \dfrac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\} \varepsilon_n$, n = 0, 1, 2, 3, … … (11)

or, $\theta_n = \left\{\dfrac{f^{/}(\eta^1{}_n) - f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\} \varepsilon_n$

we choose the initial interval $[a_0, b_0]$ so small that $f^{/}(x)$ has the same sign therein and if M_1, m_1 respectively denote the maximum and minimum of $|f^{/}(x)|$ in $[a_0, b_0]$, its oscillation $M_1 - m_1 < m_1$. Hence $|\theta_n| \le \dfrac{M_1 - m_1}{m_1} |\varepsilon_n|$ i.e. $|\theta_n| \le \left(\dfrac{M_1 - m_1}{m_1}\right)^n |\varepsilon_0|$. Since

$0 < \dfrac{M_1 - m_1}{m_1} < 1$, therefore $\theta_n \to 0$ as $n \to \infty$. This leads to $w_n \to \alpha$ as $n \to \infty$. Since ξ_n lies between α and w_n , therefore it follows that ξ_n also tends to α as $n \to \infty$. Hence from (8) it follows that $\varepsilon_n \to 0$ as $n \to \infty$ i.e. $x_n \to \alpha$ as $n \to \infty$. Therefore the method will converge.

Order of Convergence

For simplicity and without loss of generality and in view of (7) and (9), (6) can be written as

$$x_{n+1} = w_n - \frac{f(w_n)}{f^{/}(w_n)}.$$

Since α is a root of the equation f(x) = 0, therefore

$$0 = f(\alpha) = f(w_n - \theta_n) = f(w_n) - \theta_n f^{/}(w_n) + \frac{\theta_n^{\ 2}}{2!} f^{//}(c),$$

where $\min\{\alpha, w_n\} < c < \max\{\alpha, w_n\}$

or, $w_n - x_{n+1} - \theta_n + \dfrac{\theta_n^{\ 2}}{2!}\dfrac{f^{//}(c)}{f^{/}(w_n)} = 0$

or, $\alpha - x_{n+1} = -\ \dfrac{1}{2}\ \theta_n^{\ 2} \dfrac{f^{//}(c)}{f^{/}(w_n)}$

or, $\dfrac{\varepsilon_{n+1}}{\theta_n^{\ 2}} = \dfrac{1}{2}\ \dfrac{f^{//}(c)}{f^{/}(w_n)}$

or, $\dfrac{\varepsilon_{n+1}}{\left\{1 - \dfrac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\}^2 \varepsilon_n^{\ 2}} = \dfrac{1}{2}\ \dfrac{f^{//}(c)}{f^{/}(w_n)}$ [From (11)]

or, $\dfrac{\varepsilon_{n+1}}{\varepsilon_n^{\ 2}} = \dfrac{\dfrac{1}{2}\left\{1 - \dfrac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\}^2 f^{//}(c)}{f^{/}(w_n)}$

or, $$\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^{\ 2}}\right| = \frac{1}{2}\left|\frac{f^{//}(c)}{f^{/}(w_n)}\right| \left\{1 - \frac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\}^2 \qquad (12)$$

Now $0 < \left\{1 - \dfrac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\}^2 < 1$, since $f^{/}(x)$ has the same sign in $[a_0, b_0]$. Now since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all n, if iteration converge, $b_n - a_n \to 0$ and c, w_n, $\eta^1{}_n$, $\eta^2{}_n$ all are tends to α as n → ∞ and from (12) $\left|\frac{\varepsilon_{n+1}}{\varepsilon_n{}^2}\right| \to 0$ as n → ∞. This shows that the order of convergence of our new method (6-7) is more than second order. Accordingly our method is faster than the Newton's method.

Another particular method

If we consider the first two terms of the expression (5) then the approximate value of D is given by

$$D = \frac{f(w_0)}{f^{/}(w_0) + \frac{D}{2} f^{//}(w_0)} \tag{13}$$

i.e. D satisfies a quadratic equation given by, $D^2 f^{//}(w_0) + 2Df^{/}(w_0) - f(w_0) = 0$ (14)

Now again we approximated the value of D by putting $D = \frac{f(w_0)}{f^{/}(w_0)}$ in the right hand side of (13) and we get

$$D = \frac{f(w_0)}{f^{/}(w_0) + \frac{f(w_0)}{2f^{/}(w_0)} f^{//}(w_0)} = \frac{2f(w_0)f^{/}(w_0)}{2\{f^{/}(w_0)\}^2 + f(w_0)f^{//}(w_0)} \tag{15}$$

Hence the n-th approximation of the root α is given by

$$x_{n+1} = b_n - \frac{b_n - a_n}{1 - \frac{f(a_n)}{f(b_n)}} - D, \text{ if } \left|\frac{f(a_n)}{f(b_n)}\right| < 1$$

$$= a_n + \frac{b_n - a_n}{1 - \frac{f(b_n)}{f(a_n)}} - D, \text{ if } \left|\frac{f(b_n)}{f(a_n)}\right| < 1 \tag{16}$$

$$= \frac{a_n + b_n}{2} - D, \text{ if } \left|\frac{f(a_n)}{f(b_n)}\right| = 1$$

where D is given by $D = \frac{2f(w_n)f^{/}(w_n)}{2\{f^{/}(w_n)\}^2 + f(w_n)f^{//}(w_n)}$ and w_n is given by (7)

Convergence Analysis

Consider the iteration function g as expressed by Eq. (17)

$$g(x) = x - \frac{2f(x)f^{/}(x)}{2\left\{f^{/}(x)\right\}^2 + f(x)f^{//}(x)} \tag{17}$$

$$= x - \frac{f(x)}{f^{/}(x)} + \frac{f^2(x)f^{//}(x)}{2\left\{f^{/}(x)\right\}^3 + f(x)f^{/}(x)f^{//}(x)} \tag{18}$$

Executing some computation one can get the following derivatives:

$$g^{/}(\alpha) = 0 \text{ and } g^{//}(\alpha) = \frac{f^{//}(\alpha)}{f^{/}(\alpha)} + \frac{f^{//}(\alpha)}{f^{/}(\alpha)} = 2\frac{f^{//}(\alpha)}{f^{/}(\alpha)} \neq 0.$$

Hence the order of convergence is at least 2 but not greater than or equal to 3

Now to increase the order of convergence we add, $-2\frac{f^2(x)f^{//}(x)}{2\left\{f^{/}(x)\right\}^3 + f(x)f^{/}(x)f^{//}(x)}$ to the right hand side of (18) and we get the new and modified iteration function g as

$$g(x) = x - \frac{f(x)}{f^{/}(x)} + \frac{f^2(x)f^{//}(x)}{2\left\{f^{/}(x)\right\}^3 + f(x)f^{/}(x)f^{//}(x)} - 2\frac{f^2(x)f^{//}(x)}{2\left\{f^{/}(x)\right\}^3 + f(x)f^{/}(x)f^{//}(x)}$$

$$= x - \frac{f(x)}{f^{/}(x)} + \frac{f^2(x)f^{//}(x)}{2\left\{f^{/}(x)\right\}^3 + f(x)f^{/}(x)f^{//}(x)}, \text{ so that } g^{/}(\alpha) = g^{//}(\alpha) = 0 \tag{19}$$

and $$g^{///}(\alpha) = \frac{f^{///}(\alpha)}{f^{/}(\alpha)} - \frac{3\left\{f^{//}(\alpha)\right\}^2}{\left\{f^{/}(\alpha)\right\}^2} \neq 0 \tag{20}$$

and consequently the new and modified value of

$$D = \frac{f(w_n)}{f^{/}(w_n)} + \frac{f^2(w_n)f^{//}(w_n)}{2\left\{f^{/}(w_n)\right\}^3 + f(w_n)f^{/}(w_n)f^{//}(w_n)} \tag{21}$$

Now from Taylor's limited expansion of $g(w_n)$ around α, we get for $\min\{w_n, \alpha\} < \xi_n < \max\{w_n, \alpha\}$,

$$x_{n+1} - \alpha = g(w_n) - g(\alpha) = (w_n - \alpha)g^{/}(\alpha) + \frac{(w_n - \alpha)^2}{2}g^{//}(\alpha) + \frac{(w_n - \alpha)^3}{6}g^{///}(\xi_n)$$

and that according to equation (19) and (20) and for $w_n \neq \alpha$ is equivalent to $\frac{x_{n+1} - \alpha}{(w_n - \alpha)^3}$ $= \frac{g^{///}(\xi_n)}{6}$

Hence for $g^{///}(\alpha) \neq 0$ we get

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(w_n-\alpha)^3} = \frac{g^{///}(\lim_{n\to\infty}\xi_n)}{6} = \frac{g^{///}(\alpha)}{6} \neq 0 \text{ as } n \to \infty \tag{22}$$

i.e. $\left|\frac{\varepsilon_{n+1}}{\theta_n^{\ 3}}\right| = \left|\frac{g^{///}(\alpha)}{6}\right|$ as $n \to \infty$

i.e. $\lim_{n\to\infty}\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^{\ 3}}\right| = \left|\frac{g^{///}(\alpha)}{6}\right| \lim_{n\to\infty}\left|\left\{1-\frac{f^{/}(\eta^2{}_n)}{f^{/}(\eta^1{}_n)}\right\}^3\right|$

i.e. $\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^{\ 3}}\right| = 0$ as $n \to \infty$ and $\eta^1{}_n,\ \eta^2{}_n \to \alpha$ (23)

Hence finally the n-th approximation of the root α is given by,

$$x_{n+1} = b_n - \frac{b_n - a_n}{1-\frac{f(a_n)}{f(b_n)}} - D, \text{ if } \left|\frac{f(a_n)}{f(b_n)}\right| < 1$$

$$= a_n + \frac{b_n - a_n}{1-\frac{f(b_n)}{f(a_n)}} - D, \text{ if } \left|\frac{f(b_n)}{f(a_n)}\right| < 1 \tag{24}$$

$$= \frac{a_n + b_n}{2} - D, \text{ if } \left|\frac{f(a_n)}{f(b_n)}\right| = 1$$

where D is given by

$$D = \frac{f(w_n)}{f^{/}(w_n)} + \frac{f^2(w_n) f^{//}(w_n)}{2\{f^{/}(w_n)\}^3 + f(w_n) f^{/}(w_n) f^{//}(w_n)} \tag{25}$$

and from (23) it follows that, the order of convergence of the proposed method (24-25) is more than third order.

To reduce the operational count of D we may simplify D by considering

$$u_n = u(w_n) = \frac{f(w_n)}{f^{/}(w_n)}. \text{ If we consider } u_n << 1 \tag{26}$$

and on computing some steps D may be written as

$$D = \frac{f(w_n)}{f^{/}(w_n)}\left\{1+\frac{f(w_n-u_n)}{f(w_n)+f(w_n-u_n)}\right\} \tag{27}$$

in which $f^{//}(w_n)$ is absent. An alternative form of (25) is (27). Hence an alternative form of (24) and (25) is (24) and (27). The order of convergence of (24) and (27) is also more than third order [See Theorem 3.1]

Further Development

Consider the family

$$D = \frac{f(w_n)}{f^{/}(w_n)}\left\{1+\frac{f(w_n-u_n)}{f(w_n)+\lambda f(w_n-u_n)}\right\}, \text{ where } \lambda \text{ is a real parameter.} \quad (28)$$

Theorem3.1: Let f: A[a_0, b_0] → R is defined and continuous and n times differentiable in (a_0, b_0). If f(x) has a simple root α ∈ [a_0, b_0], then the family (28) with (24) has an order of convergence

i) more than third order, for $\lambda \neq -2$;

ii) more than fourth order, for $\lambda = -2$

Proof: In view of (7) and (9) in either case (24) and (28) can be written as

$$x_{n+1} = w_n - \frac{f(w_n)}{f^{/}(w_n)}\left\{1+\frac{f(w_n-u_n)}{f(w_n)+\lambda f(w_n-u_n)}\right\} \quad (29)$$

consider the iteration function ϕ as expressed by (30) as

$$\phi(w) = w - \frac{f(w)}{f^{/}(w)}\left\{1+\frac{f(w-u)}{f(w)+\lambda f(w-u)}\right\},\ u = \frac{f(w)}{f^{/}(w)}. \quad (30)$$

We denote $\varepsilon = x-\alpha$, $\theta = w-\alpha$, $\delta = \phi(w)-\alpha$, $u = u(w) = \frac{f(w)}{f^{/}(w)}$ and $c_2 = \frac{f^{//}(\alpha)}{2f^{/}(\alpha)}$

Now $f(w) = \theta f^{/}(\alpha) + \frac{\theta^2}{2} f^{//}(\alpha) + O(\theta^3)$ and $f^{/}(w) = f^{/}(\alpha) + \theta f^{//}(\alpha) + O(\theta^2)$

$$u = \frac{f(w)}{f^{/}(w)} = \frac{\theta f^{/}(\alpha) + \frac{\theta^2}{2} f^{//}(\alpha) + O(\theta^3)}{f^{/}(\alpha)+\theta f^{//}(\alpha)+O(\theta^2)}$$

$$= \left[\theta + \theta^2 c_2 + O(\theta^3)\right]\left[1 - 2\theta c_2 + O(\theta^2)\right] = \theta - \theta^2 c_2 + O(\theta^3)$$

and $u-\theta = -\theta^2 c_2 + O(\theta^3)$ i.e. $\theta - u = \theta^2 c_2 + O(\theta^3)$ (31)

$$f(w-u) = (\theta-u)f^{/}(\alpha) + \frac{(\theta-u)^2}{2} f^{//}(\alpha) + O(\theta-u)^3$$

$$\frac{f(w-u)}{f(w)}=\frac{(\theta-u)f^{/}(\alpha)+\dfrac{(\theta-u)^2}{2}f^{//}(\alpha)+O((\theta-u)^3)}{\theta f^{/}(\alpha)+\dfrac{\theta^2}{2}f^{//}(\alpha)+O(\theta^3)}$$

$$=\frac{\theta-u}{u}-(\theta-u)c_2+O(\theta^3)\ \text{[From (31)]}=\frac{\theta-u}{u}+O(\theta^2)$$

$$\frac{f(w-u)}{f^{/}(w)}=\frac{(\theta-u)f^{/}(\alpha)+\dfrac{(\theta-u)^2}{2}f^{//}(\alpha)+O((\theta-u)^3)}{f^{/}(\alpha)+\theta f^{//}(\alpha)+O(\theta^2)}$$

$$=\left[(\theta-u)+(\theta-u)^2c_2+O(\theta-u)^3\right]\left[1-2\theta c_2-O(\theta^2)\right]$$

$$=(\theta-u)-2\theta(\theta-u)c_2+O(\theta^4) \tag{32}$$

Now $\delta=\phi(w)-\alpha=w-\dfrac{f(w)}{f^{/}(w)}\left\{1+\dfrac{f(w-u)}{f(w)+\lambda f(w-u)}\right\}-\alpha$

$$=\theta-u-\frac{\dfrac{u.f(w-u)}{f(w)}}{1+\lambda\dfrac{f(w-u)}{f(w)}}=(\theta-u)-\frac{(\theta-u)-2\theta(\theta-u)c_2+O(\theta^4)}{1+\lambda\left\{\dfrac{\theta-u}{\theta}+O(\theta^2)\right\}}$$

Since λ is a real parameter and sufficiently small θ we have

$$\delta=(\theta-u)-\left[(\theta-u)-2\theta(\theta-u)c_2+O(\theta^4)\right]\left[1-\lambda\frac{\theta-u}{\theta}+O(\theta^2)\right]$$

$$=\lambda\frac{(\theta-u)^2}{\theta}-2\lambda(\theta-u)^2c_2+2\theta\left[\theta^2c_2+O(\theta^3)\right]c_2+O(\theta^4)$$

$$=\lambda{c_2}^2\theta^3+2{c_2}^2\theta^3+O(\theta^4)=(2+\lambda){c_2}^2\theta^3+O(\theta^4)$$

Again from (11) $\theta=\left\{1-\dfrac{f^{/}(\eta^2)}{f^{/}(\eta^1)}\right\}\varepsilon$

Therefore, $\delta=(2+\lambda){c_2}^2\left\{1-\dfrac{f^{/}(\eta^2)}{f^{/}(\eta^1)}\right\}^3\varepsilon^3+\left\{1-\dfrac{f^{/}(\eta^2)}{f^{/}(\eta^1)}\right\}O(\varepsilon^4)$

Hence for λ ≠ -2, the order of convergence of the method is more than third order and for λ = -2, the order of convergence is more than fourth order.

So in (24) if, we consider $D=\dfrac{f(w_n)}{f^{/}(w_n)}\left\{1+\dfrac{f(w_n-u_n)}{f(w_n)-2f(w_n-u_n)}\right\}$ (33)

where $u_n = \dfrac{f(w_n)}{f^{/}(w_n)}$ then (24) and (33) gives the fourth order iterative formula.

For different values of λ we get a families of D and in particular if λ = 1, then we get (27) and moreover if λ = ±∞ then we get (6).

Numerical Experiments and Comparison

We now compare our new methods with the Newton's method and other existing well-known methods given in [1- 4] in Table-1. All the examples are taken from the references at the end of this paper

The test functions f(x) are as follows:

$f_1(x) = x - 2 - e^{-x}$, α = 2.1200282389,

$f_2(x) = x^3 + x + 1$, α = - 0.682327803828,

$f_3(x) = \ln x$, α = 1.000000000000000, $f_4(x) = xe^{-x} - 0.1$, α = 0.111832559158962,

$f_5(x) = \dfrac{1}{x} - \sin x + 1 = 0$, α = - 0.62944648407.

Table-1 (Comparative Statement).

f(x)	Method	Initial Approx. x_0 / interval / $[a_0,b_0]$	Tolerance error(ε)	No. of Operation per iteration	No. of iteration	Obtained Solution
$f_1(x)$	(24) of[1]	2	10^{-10}	11	2	2.120028239
	Newton's Method	2		01	3	2.120028239
	New method(6 &7)	[2, 3]		05	2	2.1200282389
	New method(24&27)	[2, 3]		06	2	2.1200282389
	New method(24&33)	[2, 3]		07	1	2.1200282389
$f_2(x)$	Algorithm-I of[4]	-2.9	10^{-11}	06	4	-0.682327803828
	Newton's Method	-2.9		01	7	-0.682327803828
	New method(6&7)	[-3, 0]		05	5	-0.682327803828
	New method(24&27)	[-2, 0]		06	4	-0.682327803828
	New method(24&33)	[-3, 0]		07	2	-0.682327803828

$f_3(x)$	EXRF of[2]	[0.5, 5]	10^{-15}	10*	7	1.0000000000000000
	Newton's Method	-		01	-	Divergent
	New method(6&7)	[0.5, 5]		05	6	0.9999999999999998
	New method(24&27)	[0.5, 5]		06	4	0.9999999999999999
	New method(24&33)	[0.5, 5]		07	3	1.0000000000000000
$f_4(x)$	EXRF of[2]	[0, 1]	10^{-15}	10*	6	0.111833
	Newton's Method	0		01	5	0.111832559158962
	New method(6&7)	[0,1]		05	5	0.111832559158962
	New method(24&27)	[0,1]		06	3	0.111832559158962
	New method(24&33)	[0,1]		07	3	0.111832559158962
$f_5(x)$	(15) & (16) of [3]	-1.3	10^{-10}	07	6	-0.629446
	Newton's Method	-1.3		01	25	-0.62944648407
	New method(6&7)	[-1.3, -0.5]		05	4	-0.62944648406
	New method(24&27)	[-1.3, -0.5]		06	3	-0.62944648407
	New method(24&33)	[-1.3, -0.5]		07	2	-0.62944648407

*Up to the third order expansion of the exponential series

All computations are done by the 'C' programming language. Here we take the approximate solution, depending upon the precision (ε) of the computer. For the computer program the stopping criteria $|x_{n+1} - x_n| < \varepsilon$ is used, and when the stopping criteria is satisfied α is taken as the approximate value of the root. For all the numerical examples given in Table 1 the fixed stopping criteria (ε), as shown in the Table-1, is used.

Conclusion

In this work, we give a single general iterative formula (2) in which linear convergence is assured, if we remove the term D from the expression (2) and further if we take only the first term of (5) then the algorithm gives the order of convergence greater than 2 and consequently better than the Newton's method. Further, if we take first two terms of (5) then we get an iterative formula in which the order of convergence is greater than 3 and 4 according to the choice of a real parameter. The performance of our methods has been compared with the others well known methods and it gives equal or better result than the other existing methods an in particular for the function $f_1(x)$, our new method (24&33) has able to find out the root, correct to 10 decimal places only in a single iteration, whereas the other methods fails to do that. Of course we can give another iterative formula by considering more than first two terms of (5) and it is an open question whether this another new iterative formula can give an order of convergence greater than 4 or 5 or more.

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